WHITE'S SERIES OF MATHEMATICS

KEY TO
ELEMENTS OF GEOMETRY

JOHN MACNIE

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TO

ELEMENTS OF GEOMETRY

PLANE AND SOLID

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KEY TO MACNIE'S GEOM.
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PREFACE.

A KEY to a set of geometrical exercises is needed for much the same reasons as render it desirable to supply the teacher with answers to exercises in arithmetic and algebra. To the instructor who has skill in working out such exercises, the doing so is a mere drudgery in a task that has lost the interest of novelty; while to those who find some difficulty in the task, the time and mental exertion required is a serious addition to their necessary burdens. The unusually large number of exercises of various kinds given in the work to which this serves as a key, renders it all the more imperative to lighten the teacher's burden as much as possible.

A great proportion of the exercises, such as the questions at the end of each book and the exercises at the foot of the page, are of so easy a character as to require little more than suggestions, though even in these assistance is given proportioned to the difficulty of the case. Numerical answers are given correct to two places of decimals when not expressed as rational numbers. The number of exercises at the end of each book is still so large as to preclude the possibility of giving a full-length demonstration of each, if the book is to be kept within reasonable limits of size and price. It will be found, however, that all needful information is given as concisely as is consistent with clearness, the diagrams, when not given, being easy to draw, according to directions, on a plan explained in the Introduction. In this Introduction will also be found a general discussion of the difficulties to be met in the solution of geometrical exercises, and how best to overcome them.

KEY TO GEOMETRY.

INTRODUCTION.

THE first and indispensable requisite towards facility in the solution of geometrical exercises is familiarity with the propositions from which their solutions may be deduced. It is chiefly such a familiarity that enables the practiced geometrician to see at a glance upon what propositions an exercise probably depends that may present considerable difficulty tc a beginner. The latter frequently fails to perceive the short and easy path to a result which he reaches, perhaps, after taking a very circuitous route. This difficulty is obviated to a great extent when the exercise is given as a *rider;* that is, a proposition to be deduced mainly from a given proposition, as is the case with the exercises at the end of Book I. While this relieves the beginner from his most formidable difficulty, it must be remembered that a given exercise depends seldom, if ever, upon a single proposition, but usually also upon others with which the mind has to be familiar in order to see this dependence. The majority of the propositions of Book I. are of such importance that pains should be taken to make the truths they express as present to the mind as those of the multiplication table, advantage being taken of such aids to mental association as may prove available. Thus the first five propositions, stating the general properties of angles, form an easily remembered group by themselves. The general propositions concerning equal triangles may be associated with the numbers 63, 66, 69; the two cases of equal right triangles with 72 and 73; the three double propositions concerning parallels with 110, 112, 113; and so on. It is largely in order to promote this all-important familiarity with leading principles that so many easy exercises are given in the course of each book, and questions at its end. The latter may, according to circumstances, be taken *pari passu* with the progress of the class through the book, or be left for review.

Assuming the student's mind to be duly prepared, as adverted to in the preceding paragraph, the first step toward the solution of a given exercise is carefully to read over its wording so as to have a clear idea of what is wanted. Proper care in doing this will not only save misdirected effort, but will also probably suggest some notion of how to set about the solution.

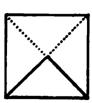
What has been thus far said applies equally to all exercises, whether theorems or problems; but, in what immediately follows, the solution of theorems only will be discussed, leaving the special consideration of problems for another place.

Solution of Theorems.

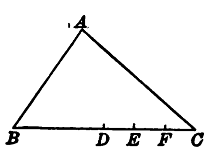
The first general direction, given on p. 72 of the Geometry, reads as follows:

1. *Assuming the theorem as true, construct a diagram accordingly.*

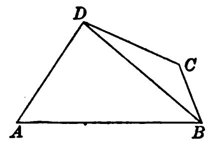
It may seem, at first sight, a comparatively easy matter to draw the diagram for a theorem, seeing that it is given in words, at least. Yet there are some points worthy of attention, even in this seemingly easy matter. There are things to be avoided and timesaving devices of which use can be made. We must, in the first place, avoid giving the figure any special characteristics not called for in the statement. If, for example, the figure referred to in the statement is simply a triangle, neither right nor obtuse, nor isosceles nor equilateral, we must take care not to make our triangle have any of these characteristics,



since otherwise we may obtain a misleading diagram suitable only for some special, and possibly easy, case of the theorem under discussion. The Pythagorean theorem, for example, admits of a very simple proof when the arms of the right triangle are equal. What we require, in general, is what may be called a *neutral* triangle. Such a triangle may be constructed

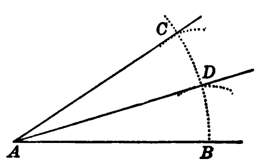


on any given base BC as follows. Taking by inspection the mid point D of BC, we roughly divide DC into three equal parts in E and F; then arcs described from B and C as centers, with radii equal to BE, BF respectively, will give the vertex of the required neutral triangle ABC; which, of course, can be described offhand, when once the form is familiar to the eye. The neutral triangle, again, can be made the basis of the neutral quadrilateral $ABCD$. It is to be understood



hereafter, that, whenever a triangle ABC or a quadrilateral $ABCD$ is referred to without a diagram, the triangle or quadrilateral is of the form and lettering of those given above.

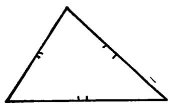
In the actual construction of our diagram, again, we may save ourselves much useless expenditure of time by performing the construction, not in the order of the statement, but in the order most convenient for the production of the required figure. If, for example, a side of a triangle is given in the statement as trisected, we do not first construct the triangle and then laboriously proceed to trisect a side, but we first lay off on a line three parts of convenient length, and then employ this three-part line as may be required in the construction of our figure. If we require a bisected angle BAC, we may describe from A as center any convenient arc BC, on which we lay off equal parts BD, DC; etc. If we require a quadrilateral whose diagonals intersect in a given manner, whether in regard to the length of the segments or to the angle they make, we first draw the diagonals according to specification, then complete the quadrilateral. If we require a circle inscribed in, or circumscribed about, a polygon, we first describe the circle, then construct the polygon without or within, as the case may be.



For beginners, at least, it is highly desirable that they construct their diagrams carefully. The practiced geometrician may do well enough with roughly sketched diagrams that have little resemblance to the objects they are supposed to represent; but to suggest this to the beginner would be about as reasonable as to teach a child reading by the medium of difficult manuscript. On the principle of "One difficulty at a time," the beginner should so construct his diagram that it shall be a help rather than a hindrance, not suggesting relations that do not exist, but rather inviting attention to such as might otherwise escape his attention,—as that certain lines are equal, or parallel, or perpendicular, etc. If, by deduction from the given relations of the figure, he finds that these suggested relations are real, he has probably a clew to the course of reasoning to be followed in the solution.

In order to have carefully constructed diagrams with the least expenditure of time, the student may have recourse to the following device, often found of great advantage by the writer and others to

whom he has suggested it. On pieces of cardboard,* construct carefully in ink, on a scale about four or five times as great as that of the corresponding figures in the text-book, an equilateral triangle, an isosceles triangle with arms nearly double the base, a neutral triangle,



a parallelogram, or other frequently required figure. On each side of the triangle mark its mid point and the point where the bisector of the opposite angle meets it. If, then, the exercise involves the bisection of an angle or angles of a triangle, we take the prepared triangle, draw in pencil the bisector, and have at once an accurate and suggestive diagram of convenient size, having in a permanent form the parts of most importance and requiring most time to prepare. One such permanent figure has been used as the basis of nearly a hundred diagrams.

Supposing the diagram prepared, we come to the most difficult part of our task, the difficulty being of much the same nature as that which confronts us in framing the equation for the solution of an algebraic problem, though greater in degree, since the directions that can be given are of an even more general character. If we proceed by the method of analysis, the direction, as given on p. 72 of the Geometry, reads:

2. *Deduce, with the aid of such constructions as may be necessary, any consequences that follow from the assumption that the theorem is true, applying such theorems already proved as are applicable to the diagram.*

Objection has been made to the use of the terms *analysis* and *synthesis* as applied to the method of procedure here referred to, and that employed in the demonstration. This use of the terms is, however, of long standing. Perhaps the appellations *ingressive* and *regressive reasoning* would more aptly designate a distinction that consists rather in a difference of direction than of method. In the ingressive, or so-called *synthetic*, process we reason *forward* from premises known to be true, and if our reasoning is correct, all our deductions will be equally true. In the regressive, or so-called *analytic*, process we reason *back* from premises only assumed as true, deducing consequence after consequence until we reach one that we know to be true or false. The method of reasoning is really the same in both cases. The difference lies in the nature of its basis and in the object aimed at, this being, in the analytic method, as before said, to arrive at some

* The covers of old scratch books, usually about 6 × 8, are very suitable for this purpose, affording space on each side for two such figures as those suggested.

result that we *know* to be true or false, in order to make this result the basis for the proof or disproof of the assumed theorem.

The question arises, *Is it always necessary to begin the solution of a given exercise by this analytic process?* Certainly not, as regards theorems, at least, though it is usually necessary with problems. Most of the theorems given as exercises are of such a simple character that, once they are clearly understood and the diagram prepared, their dependence upon some known theorem is obvious, and we can proceed at once to the synthetic proof. At the same time it may be recommended, as an advantageous training for the attack of more difficult exercises, to take, in every case, as many steps in the analysis as will make the synthetic basis obvious.

Thus in Ex. 112, though it is evident that the theorem is merely a special case of Prop. IV., yet, designating the prolongation of BA as AD, we may reason thus: *If CAB, CAD are both right angles, BAD is a straight line. But BAD* is *a straight line* (Hyp.); etc.

In Ex. 120, if ABC be the triangle, and the bisectors meet in O, it is seen at once that since $\angle OBC = \angle OCB$, being halves of equal angles, it follows as an immediate consequence that $OC = OB$. Yet we may reason thus: *If $OC = OB$, then $\angle OBC = \angle OCB$; but these angles* are *equal, being halves of equal angles*; etc.

In Ex. 147 the dependence is not obvious, so we proceed as follows:

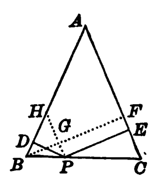
$$\text{If } \angle D = \tfrac{1}{2}\angle A,$$
$$\text{then } \tfrac{1}{2}\angle ABC + \angle D = \tfrac{1}{2}\angle ABC + \tfrac{1}{2}\angle A.$$

(We make this addition to both members of the equality because, supposing BC produced to E, we perceive that $\angle DCE$ or $\frac{1}{2}\angle ACE = \angle DBC$ or $\frac{1}{2}\angle ABC + \angle D$, while $\angle ACE = \angle ABC + \angle A$). (122)*

Examination of the diagram showing us that this equality is true, we proceed to the synthesis:

$$\left.\begin{array}{l}\text{Since } \angle DCE = \angle DBC \text{ or } \tfrac{1}{2}\angle ABC + \angle D,\\ \text{and } \angle DCE \text{ or } \tfrac{1}{2}\angle ACE = \tfrac{1}{2}\angle ABC + \tfrac{1}{2}\angle A,\end{array}\right\} \quad (122)$$
$$\tfrac{1}{2}\angle ABC + \angle D = \tfrac{1}{2}\angle ABC + \tfrac{1}{2}\angle A; \ \therefore \ \angle D = \tfrac{1}{2}\angle A.$$

Thus far we have needed no addition to the original diagram.



In Ex. 149 let PD, PE be the distances from a point P in the base to AB, AC respectively. Draw the altitude BF to AC.

If $PD + PE = BF$, as the theorem asserts, then by cutting off a part of BF equal to PE, as FG, we obtain $GB = PD$. We evidently can obtain the point G by

* The reference is to Art. 122 of the Geometry.

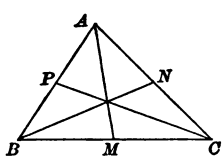
drawing PG parallel to AC, since then PF will be a parallelogram; and, producing PG to meet AB in H, we have an isosceles $\triangle HBP$, since $\angle HPB = \angle C = \angle B$. Then as PD is perpendicular to AB (Hyp.), and BG is perpendicular to PH, being perpendicular to its parallel AC, we have the altitudes to the arms of isosceles $\triangle HPB$ equal. Now this is true by Ex. 119; hence the synthesis:

Draw $PH \parallel$ to AC, to meet the altitude BF in G, and AB in H.

Since $\angle HPB = \angle C$ (112) $= \angle B$, $\triangle HPB$ is isosceles; and as PD is perpendicular to HB (Hyp.), and BG is perpendicular to PH, being perpendicular to AC, $PD = BG$ (Ex. 119), and PF being a parallelogram by construction, $PE = GF$; $\therefore PD + PE = BG + GF = BF$.

Special attention should be paid to the foregoing analysis, as it affords an example of what almost always occurs, unless the analysis is of the simplest character; that is, the mingling with it of a certain amount of synthesis. Thus the second sentence of the analysis, that beginning, "We evidently," is wholly synthetic in character, being quite independent of the assumed theorem. This combination of the two methods is that most practically useful. Whenever we make an addition to the original diagram, such as drawing a parallel to some line, we have to develop the relations of such lines synthetically in order to help out our analysis; while, again, in our synthetical reasoning, if we come to a result that seems as if it *ought* to be true, though we are not able to say that it is so, we may help ourselves by inquiring, *If this were true, what consequences would result from it?*

In conclusion, it may be remarked that the analytic method is best suited to exercises in which special results or equalities are to be established, as in the foregoing examples. It is much less suited to exercises in which general results or inequalities are to be established. Thus if, in Ex. 142, we let AM, BN, CP, be the medians from A, B, C, resp., we will find that to assume the inequality $AM + BN + CP < AB + AC + BC$, leads to the desired result by a somewhat roundabout way. Slight study of the diagram will, on the other hand, suggest the synthesis



$AM < \frac{1}{2}(AB + AC)$, $BN < \frac{1}{2}(AB + BC)$, $CP < \frac{1}{2}(AC + BC)$; (Ex. 138)

$$\therefore AM + BN + CP < \tfrac{1}{2}(2\,AB + 2\,AC + 2\,BC) < AB + AC + BC.$$

Similarly, for the second part of the theorem,

$$AM > \tfrac{1}{2}(AB + AC - BC),\quad BN > \tfrac{1}{2}(AB + BC - AC),$$

$$CP > \tfrac{1}{2}(AC + BC - AB);$$

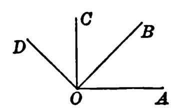
$$\therefore AM + BN + CP > \tfrac{1}{2}(AB + AC + BC).$$

Little need be said about the third general direction, as it may reasonably be assumed that only true theorems will be given as exercises, yet it finds an important application whenever, as before adverted to, we arrive at some suggested relation which we do not see how to prove. We assume the thing to be true, and trace what consequences would result from it. If we are thus led to a contradiction of some established truth, or to something inconsistent with the data of the given question, we know that the suggested relation cannot be true, and reject it accordingly.

QUESTIONS, p. 17.

(Only slight hints are given of the answers to most of these questions.)

1. $AB = AC + CB$; $AC = AB - CB$. **2.** A straight line. **3.** A right angle. **4.** A straight line. **5.** A curved line. **6.** Plane surfaces, which are also presented by the surface of still water, of a pane of glass, etc. **7.** By applying the straightedge to the given surface and twisting it about in order to ascertain whether the edge always coincides with the given surface. Mechanics often oil the surface that is to be tested in this way. Try this on a slate. **8.** The surface of a stovepipe is not a plane, because straight lines can be drawn on it only in a certain direction, and not between *any* two points. Illustrate the same thing on a conical surface. **9.** Only a curve can be drawn on the surface of an eggshell. Illustrate the same thing on the surface of a round pebble or apple. **10.** The angles are AOB, BOC, COD, AOC, AOD, BOD. Of these, AOB and BOC are adjacent, as are also AOC and COD, BOC and COD. AOC is the sum of two angles, AOB and BOC, while AOD is the sum of AOB, BOC, and COD.



11. Yes; in the preceding figure AOB and AOC have the common vertex O and the common side OA, yet are not adjacent. **12.** An acute angle is less than, a right angle is equal to, an obtuse angle is greater than, its supplement.

QUESTIONS, p. 21.

1. See p. 20. **2.** We first draw a straight line nearly as long as the straightedge, then shift the latter so as to coincide with only a part of the line just drawn. The part of the line now drawn will have the same direction as the first part (15). **3.** We open the dividers or compasses so that the points coincide with the extremities of CD. If we then place one of the points on an extremity of AB, the other point will mark off a distance equal to CD (20). **4.** BE is laid off

equal to CD on AB produced, as shown in the preceding answer. Then $AE = AB + BE = AB + CD$ and $BE = AE - AB$. **5.** An obtuse angle. **6.** An acute angle. **7.** The angles are BAC, CAD, and BAD. **8.** Confining ourselves to one side of BAE, we have the six angles, BAC, CAD, DAE, BAD, CAE, BAE. **9.** *The sum of the angles BAC and CAD is equal to the angle BAD.* **10.** *The difference of the angles BAD and BAC is equal to the angle CAD.* **11.** *The angle BAD is greater than the angle CAD.* **12.** *The angle BAC is less than the angle BAD.* **13.** $AB + BC = AC$. **14.** $AC - BC = AB$. **15.** $AC > BC$. **16.** $AB < AC$. **17.** $\angle BAD + \angle BAC$ is equal to a right angle. **18.** $\angle BAD - \angle BAC = \angle CAD$.

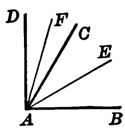
EXERCISES, pp. 24–68.

1. Let $\angle A$ denote the angle, and $\frac{1}{2}\angle A$ its complement. Then

$$\angle A + \tfrac{1}{2}\angle A = \text{a rt. } \angle; \therefore \angle A = \tfrac{2}{3} \text{ rt. } \angle = \tfrac{1}{3} \text{ st. } \angle.$$

2. Let $\angle A$ denote the angle, and $\frac{1}{3}\angle A$ its supplement. Then

$$\angle A + \tfrac{1}{3}\angle A = \text{a st. } \angle; \therefore \angle A = \tfrac{3}{4} \text{ st. } \angle = \tfrac{3}{2} \text{ rt. } \angle.$$



3. Let AE, AF be the bisectors of the adjacent complementary angles CAB, CAD.

$$\angle CAB + \angle CAD = \text{a rt. } \angle;$$
$$\therefore \tfrac{1}{2}\angle CAB + \tfrac{1}{2}\angle CAD = \tfrac{1}{2} \text{ rt. } \angle;$$
$$i.e., \ \angle CAE + \angle CAF = \tfrac{1}{2} \text{ rt. } \angle.$$

4. $\angle EAD$ is complement of $\angle EAC$, and $\angle EAC$ of $\angle EAD$.

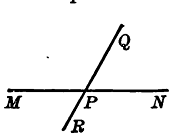
5. $\angle EAB$ is supplement of $\angle EAD$, $\angle EAD$ of $\angle EAB$, and $\angle CAD$ of $\angle CAB$, while $\angle CAB + \angle EAD$ is the supplement of $\angle CAE$.

6. For EAb being a st. $\angle$ and CAB a rt. $\angle$, $BAb + CAE =$ a rt. $\angle$.

7. If $\angle BAC + \frac{1}{2}\angle BAC =$ a st. $\angle$, then $\angle BAC = \frac{2}{3}$ st. $\angle = \frac{4}{3}$ rt. $\angle$, and $\angle CAD$ or $\frac{1}{2}\angle BAC = \frac{1}{3}$ st. $\angle = \frac{2}{3}$ rt. $\angle$.

8. For $\angle AOB + \angle AOD = \angle DOC + \angle BOC =$ a st. $\angle$; that is, $\angle$s AOB and AOD are supplementary, as are also $\angle$s DOC and BOC.

9. In order that BA may be in a straight line with AD, $\angle BAD$ must be a st. $\angle$; or, by the conditions of the question, $\angle BAC + \frac{7}{9}$ rt. $\angle$ must equal two rt. $\angle$s; *i.e.*, $\angle BAC = \frac{11}{9}$ rt. $\angle$.



10. If $\angle QPM = \angle RPN$, then $\angle QPN = \angle RPM$ (Ax. 3); whence

$$\angle QPM + \angle RPM = \angle QPM + \angle QPN = \text{a st. } \angle;$$
$$\therefore QR \text{ is a straight line.}$$

11. The angles A, A' are the supplements of the angles formed by producing A, A' in both directions, and the angles thus formed are equal (44).

12. The exterior angles at C are equal, being the supplements of angles ACB, $A'CB'$, which have been proved equal.

13. For then it would have two perpendiculars drawn from the same point to the same line, which is impossible (51).

14. For these exterior angles are the supplements of equal angles.

15. They cannot be right angles, by Ex. 13, much less can they be obtuse. The third angle may be acute, right, or obtuse.

16. Since the interior angles are unequal, the exterior angles that are respectively supplementary to them must be unequal.

17. For each of the two exterior angles thus formed is a right angle.

18. For each of the two exterior angles thus formed, being the supplement of an obtuse angle, must be acute.

19. For A will coincide with A', while O retains its position.

20. Since $\angle s\ AOC$, $A'OC$ can be made to coincide, they are right angles (30). The bisection of $\angle s\ A$ and A', and that of BC, can be proved by an anticipation of the demonstration of Prop. VIII., by making $\angle ACA'$ coincide with $\angle ABA'$; etc.

21. For this we assume that $OB = OC$ has been proved, as in Ex. 20.

22. $AO = A'O$, OC is common, and $\angle AOC = \angle A'OC$; etc.

23. In the first case AC would fall without $\angle A'$; in the second, AC would fall within $\angle A'$.

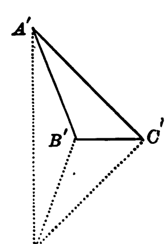
24. (1) The angles opposite unequal sides cannot be equal, since then the sides opposite them would be equal. (2) The sides opposite unequal angles cannot be equal, since then the angles opposite them would be equal.

25. For if it had two equal angles, it would have two equal sides, and would not be scalene.

26.
$$\angle ABC = \angle ACB,\ \angle ABA' = 2\angle ABC,$$
$$\text{and } \angle ACA' = 2\angle ACB; \text{ etc.}$$

27. Since $AB = AC$, $OB = OC$, and AO is common,
$$\triangle AOB = \triangle AOC; \text{ etc.}$$

28.
$$AB' = A'B' \text{ and } AC' = A'C';$$
$$\therefore \angle C'AA' = \angle C'A'A,\ \angle B'AA' = \angle B'A'A;$$
$$\therefore \angle C'AB' = \angle C'A'B' \text{ (Ax. 3); etc.}$$



29. Since the three sides of the one are respectively equal to the three sides of the other, the triangles are equal (69) and must coincide if they are on the same side of the common base.

30. Place the triangles as in the left-hand figure of Art. 74, and join the vertices. Then the halves of the equal vertical angles being equal (74), the sides opposite those equal angles are equal; hence the triangles are equal

(69), and must coincide if placed on the same side of the common base.

31. The angles ABA', ACA' can easily be shown to be the sums, or the differences, of equal angles.

32. The exterior angles ABD, ACE are equal (44), and the including sides are equal (Hyp. and Const.); hence $\triangle ABD = \triangle ACE$; etc.

33. The interior angles adjacent to the equal exterior angles are equal (44); hence the opposite sides are equal.

34. This is a special case of Prop. VIII.

35. This is a special case of Prop. VI.

36. Since $AD=AC$, and $BE=BC$, $AD+AB+BE=AC+AB+BC$.

37. $\angle CAD = \angle CBE$, CA, AD, $=$ CB, BE respectively; etc. Or show, as in Ex. 32, that $CD = CE$; etc.

38. $AG = \frac{1}{2} AB$ (75), $GE = GB + BE = AG + 2\,AG = 3\,AG$.

39. Join AB and produce AB to C so that $AC = 2\,AB$. With A and B as centers, and radius equal to AC, describe circumferences intersecting in X; then $AX = BX = 2\,AB$.

40. Bisect AB in D, then DC in E, and proceed with AE as with AC in Ex. 39.

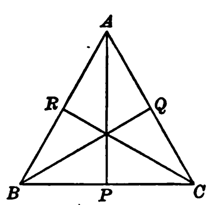
41. Since the line drawn through X and Y is perpendicular to AB at its mid point (75), it must coincide with the perpendicular drawn to AB at that point (41).

42. $\angle BDE = \angle CED$, they being supplements of equal angles ADE, AED.

43. After bisecting $\angle ABC$ by BF (81), bisect each of the angles ABF, CBF.

44. F might lie between A and DE, might coincide with A, might be in the prolongation of FA. If F coincides with A the one point cannot determine the required bisector.

45. AE, EF would be equal to AD, DF respectively, and AF be common; hence the $\triangle$s AEF, ADF would be equal, and $\angle FAE = \angle FAD$ (70).



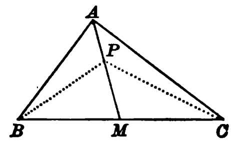
46. The $\triangle$s APB, APC, BQA, BQC, CRA, CRB, are equal (66); hence $AP = BQ$, etc.; $\angle APB = \angle APC$, etc.; $\angle BAP = CAP$, etc.

47. Proceed as in Art. 83.

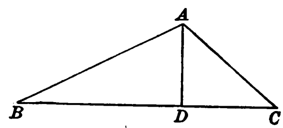
48. In the first case, the perpendicular cannot fall without the triangle, since then an acute angle would be greater than a right angle (83). In the second case the perpendicular cannot fall within the triangle, since then a right angle would be greater than an obtuse angle.

49. Proof much the same as in Art. 86.

50. Since the greatest side lies opposite the greatest angle, the angles that include the greatest side must be acute; hence by the first case of Ex. 48, the perpendicular must fall within the triangle.



51. Since BC is the greatest side, the perpendicular to BC falls within the triangle (Ex. 48); then $AB > BD$ and $AC > DC$; $\therefore AB + AC > BC$.



52. Let AM be the median drawn to BC. Since BM, $MA = CM$, MA respectively, but $AC > AB$, $\angle AMC > \angle AMB$ (91). Join any point P between A and M with B and C. Then BM, $MP = CM$, MP respectively, but $\angle AMC > \angle AMB$; $\therefore PC > PB$ (90).

53. At the given point make a right angle (95) and bisect it (81).

54. From the given point P, draw a perpendicular PQ to the given line (92), and at P draw PS, making half a right angle with PQ (Ex. 53). Let PS meet the given line in S; PSQ is the required angle (123).

55. $\angle s$ CPA, CPB are the supplements of equal $\angle s$ APD, BPD.

56. The distance of P from A or B would have to be equal to AB.

57. Since only two equal lines can be drawn from the point to the given line, there can be only two that make equal angles with it.

58. The two oblique lines form with the given line a triangle, of which the greater side is opposite the greater angle.

59. CD may be regarded as the bisector of the st. $\angle ADB$.

60. For, if it were equally distant from the sides, it would be in the bisectors of the angle (101″).

61. *Any point unequally distant from the sides of an angle is not in the bisector of that angle.* For if it were in the bisector, it would be equidistant from those sides.

62. The perpendicular forms with the sides of the angle and its bisector two triangles that are equal (63); hence the angles opposite the bisector are equal (70).

63. *Any line that makes equal angles with the sides of a given angle is perpendicular to its bisector.* For it forms with the sides and bisector of the angle two equal triangles (63).

64. (1) If the sides containing the angle are equal, the bisector is perpendicular to the third side at its mid point (77). (2) If the sides are unequal, the bisector cannot pass through the mid point of the third side, since then by drawing perpendiculars from that point to those sides, they could be proved equal, as the perpendiculars are equal.

65. If PF, QE be drawn, $\triangle POF = \triangle QOE$ (66);

$$\text{whence } \angle QPF = \angle PQE, \text{ and } PF \text{ is } \parallel \text{ to } QE.$$

66. The three equal angles are together equal to a straight angle (120), hence each is equal to one third of a straight angle.

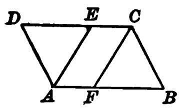
67. Let $\angle A$ be the vertical angle, B and C the equal base angles. Then $\angle A + \angle B + \angle C = \angle A + 2\angle B = 2\angle A = 2$ right angles; $\therefore$ $\angle A$ is equal to a right angle.

68. With the same symbol as above for the angles, we have

$$\angle A + \angle B + \angle C = \angle A + 2\angle B = \angle A + 2n\angle A = (2n+1)\angle A = \text{a}$$

straight angle; $\therefore$ $\angle A = \frac{1}{2n+1}$ straight angles.

69. If $\angle A > \angle B + \angle C$, $2\angle A > \angle A + \angle B + \angle C > 2$ right angles; $\therefore$ $\angle A >$ a right angle. If $\angle A < \angle B + \angle C$, we show similarly that $\angle A <$ a right angle.



70. Let the bisectors of $\angle$s A and C cut CD, AB in E, F respectively. Since $\angle EAB = \angle ECF$ (Ax. 7), and $\angle ECF = \angle CFB$ (110), $\angle CFB = \angle EAB$; $\therefore$ CF is parallel to EA (112).

71. Let the diagonals AC, BD bisect each other in O. Then $\triangle AOD = \triangle COB$; $\therefore$ $AD = BC$ and $\angle OAD = \angle OCB$; $\therefore$ AD is parallel to BC; $\therefore$ $ABCD$ is a parallelogram (142).

72. If $\angle A$ is supplementary to $\angle B$, then AD is parallel to BC; and if $\angle B$ is supplementary to $\angle C$, then AB is parallel to CD, and $ABCD$ is a parallelogram.

73. If $\angle A = \angle A'$, then also $\angle B = \angle B'$ (113, 44); hence $\angle C = \angle C'$, and $\angle D = \angle D'$ (136).

74. Let the equal diagonals intersect in O. Since $OA = OB = OC = OD$, and $\angle OAB = \angle OBA$, also $\angle OAD = \angle OCB = \angle OBC$, $\angle BAD = \angle ABC =$ a right angle; etc.

75. If the diagonal AC bisects $\angle$s A and C, so that $\angle BAC = \angle BCA$, then $BA = BC$; etc.

76. For then the opposite angle is a right angle (136), whence the remaining angles are also right angles (113).

77. Let the diagonals AC, BD intersect in O. Since $OA = OC$, $OB = OD$ (146), and $\angle AOB = \angle DOC$, $\triangle AOB = \triangle COD$; etc.

78. Let the bisectors of $\angle$s A and B meet in O. Since $\angle A + \angle B = 2$ rt. $\angle$s, $\angle OAB + \angle OBA =$ a rt. $\angle$; $\therefore$ $\angle AOB =$ a rt. $\angle$ (121).

79. The figure formed will be a quadrilateral whose diagonals bisect each other, hence it is a parallelogram (Ex. 71). If AB is perpendicular to CD, the parallelogram will be equilateral.

80. Since $AC = AD = CB = DB$ (Const.), the figure formed is equilateral; and if $EA = EC$, the figure is a square (Ex. 74).

81. A square or a rhombus, according as the vertical angle is or is not a right angle.

82. For $\frac{6-2}{6} = 2\left(\frac{3-2}{3}\right)$, employing formula of Art. 127.

83. An interior angle $= \frac{n-2}{n}$ straight angles, an exterior angle $= \frac{2}{n}$ straight angles; hence, by the conditions, $\frac{n-2}{n} = \frac{1}{n}$; $\therefore\ n = 3$.

84. If n be the number of sides of the polygon, an interior angle $= \frac{n-2}{n}$ straight angles, and an exterior angle $= \frac{2}{n}$ straight angles; hence, by the conditions, $\frac{n-2}{n} = \frac{4}{n}$; $\therefore\ n = 6$.

85. Since $\triangle ABD = \triangle CBD$ (140), and $\triangle BOE = \triangle DOF$, $OEAD = OFCB$ (Ax. 3).

86. $\triangle ABC$ can be superposed on its equal $\triangle DCB$; then as $\triangle CEO$ will coincide with its equal $\triangle BFO$, $OEAB$ will coincide with $OFDC$.

87. Draw BD cutting FH in O. $BF = DH$ (149), $\angle OBF = \angle ODH$, and $\angle OFB = \angle OHD$ (110); $\therefore\ OF = OH$, and $OB = OD$ (63); hence the diagonal CG will also pass through O (146).

88. If $AD = BC$, then $AD = DG$ (136), then $\angle A = \angle DGA = \angle B$.

89. The $\triangle$s CBK, DAG are easily proved equal, whence $BK = AG$. Now $AK + GB = 2\,DC = AB + GK$; $\therefore\ GK = 2\,DC - AB$.

90. $BQ = \frac{1}{2}BF = \frac{1}{4}BC$ (149). Also $PQ = \frac{1}{2}(AB + EF)$

$$= \tfrac{1}{2}\{AB + \tfrac{1}{2}(AB + CD)\} = \tfrac{1}{2}(\tfrac{3}{2}AB + \tfrac{1}{2}CD) = \tfrac{1}{4}(3\,AB + CD).$$

QUESTIONS, p. 69.

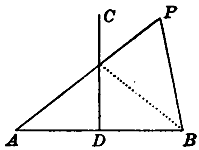
91. Such a triangle is rigid because its three sides determine a triangle (69); *i.e.*, the sides being given, the angles are also given. But any polygon of more than three sides can be *deformed*, that is made to change form while the length of its sides remains unchanged. This principle, so important in mechanical construction, may be illustrated by means of the wooden frame of a slate or similar contrivance.

92. No; illustrate by drawing an intercept parallel to any side of a given triangle.

93. (1) Erect a perpendicular to either side at the vertex. (2) Produce either side.

94. By a right angle.

95. (1) 54°, 45°, 0°. (2) 144°, 135°, 90°.



96. A reflex angle can be bisected by the process given in Art. 81.

97. Yes, with the annexed diagram, P being the point.

98. Not unless they are in the same plane. Illustrate by the nonparallel edges of a room.

99. $180° - 23° = 157°$.

100. $90°$; *i.e.*, it is a right angle.

101. $180° - 50° = 130°$, and one half of $130°$ is $65°$.

102. The base angles together $= 45° \times 2 = 90°$; hence the vertical angle $= 180° - 90° = 90°$.

103. Let $\angle A$ denote the vertical angle; then

(1) $\angle A + \angle B + \angle C = \angle A + 2\angle A + 2\angle A = 180°$;

$$\therefore\ \angle A = 36°,\ \angle B = \angle C = 72°.$$

(2) $\angle A + \angle B + \angle C = \angle A + 3\angle A + 3\angle A = 180°$;

$$\therefore\ \angle A = 25\tfrac{5}{7}°,\ \angle B = \angle C = 77\tfrac{1}{7}°.$$

(3) $\angle A + \angle B + \angle C = \angle A + n\angle A + n\angle A = 180°$;

$$\therefore\ \angle A = \frac{180°}{2n+1},\ \angle B = \angle C = 180° \times \frac{n}{2n+1}.$$

104. $3\angle A = 180°$; $\therefore\ \angle A = 60°$.

105. Let $\angle A$ and $\angle B$ denote the acute angles; then

(1) $\angle A + \angle B = \angle A + \tfrac{5}{7}\angle A = 90°$; $\therefore \angle A = 52°\ 30'$, $\angle B = 37°\ 30'$.

(2) $\angle A + \angle B = \angle A + \dfrac{m}{n}\angle A = 90°$; $\therefore \angle A = 90° \times \dfrac{n}{m+n}$,

$$\angle B = 90° \times \frac{m}{m+n}.$$

106. *Polygon, quadrilateral, parallelogram, rectangle, square.*

107. Through the intersection of the diagonals (145).

108. Twenty-five feet (148).

109. *a.* $(4-2)\ 180° = 360°$. *b.* $(5-2)\ 180° = 540°$.
c. $(6-2)\ 180° = 720°$. *d.* $(n-2)\ 180°$.

110. *a.* $\tfrac{1}{4}(4-2)\ 180° = 90°$. *b.* $\tfrac{1}{5}(5-2)\ 180° = 108°$.
c. $\tfrac{1}{6}(6-2)\ 180° = 120°$. *d.* $\tfrac{1}{10}(10-2)\ 180° = 144°$.

111. Let $\angle A$ and $\angle B$ denote the angles; then

(1) $\angle A + \angle B = \angle A + 2\angle A = 180°$; $\therefore\ \angle A = 60°$, $\angle B = 120°$.

(2) $\angle A + \angle B = \angle A + \dfrac{m}{n}\angle A = 180°$; $\therefore \angle A = 180° \times \dfrac{n}{m+n}$,

$$\angle B = 180° \times \frac{m}{m+n}.$$

THEOREMS, p. 74.

112. Let the prolongations of BA, CA be AD, AE respectively. Since BD is a straight line, BAD is a st. $\angle$, and BAC being a rt. $\angle$ (Hyp.), CAD is also a rt. $\angle$ (31); whence also DAE and BAE are rt. $\measuredangle$ (50).

113. Since the sum of the supplementary angles is a straight angle or two right angles, the sum of their halves is a right angle.

114. If the sum of the halves of two adjacent angles is a right angle, the sum of those angles must be 2 right angles; *i.e.*, they are supplementary.

115. Let the intersecting lines be AB, CD, and the bisectors of the vertical $\measuredangle AOC$, BOD be OE, OF resp. $\angle AOE = \angle BOF$; $\therefore \angle AOE + \angle AOF = \angle BOF + \angle AOF =$ a st. $\angle$; $\therefore$ EOF is a straight line.

116. Take the same diagram as above, but suppose EF to be a straight line bisecting $\angle AOC$. Then $\angle AOE = \angle BOF$ (50), and $= \angle COE$ (Hyp.); but $\angle COE = \angle DOF$; $\therefore$ $\angle BOF = \angle DOF$; *i.e.*, EF bisects $\angle BOD$.

117. Let BD, CE be the intercepts drawn to AB, AC of isosceles triangle ABC. $\angle ABD = \angle ACE$ (Ax. 7), $\angle A$ is common, and $AB = AC$; $\therefore \triangle ABD = \triangle ACE$; $\therefore BD = CE$.

118. Let DE be $\perp$ at F to AF, the bisector of $\angle BAC$, and meet the sides in D, E respectively. Since $\angle DAF = \angle EAF$, rt. $\angle AFD =$ rt. $\angle AFE$, and AF is common, $\triangle AFD = \triangle AFE$; $\therefore AD = AE$.

119. Let BD, CE be perpendiculars to AC, AB respectively, of isosceles $\triangle ABC$. Demonstrate as in Ex. 118, or by means of Prop. XII.

120. Let the bisectors of $\measuredangle B$ and C of isosceles triangle ABC meet in O. Since $\angle B = \angle C$, $\angle OBC = \angle OCB$ (Ax. 7); $\therefore OC = OB$.

121. Let BD, CE be the medians to the equal sides AC, AB of isosceles triangle ABC. Then $BE = CD$ (Ax. 7), BC is common, and $\angle B = \angle C$; $\therefore \triangle BCD = \triangle CBE$; etc.

122. *An intercept cutting off equal parts upon the sides of a given angle is perpendicular to the bisector of the angle.*

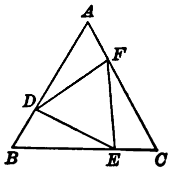
Let DE cut off equal parts AD, AE on the sides of $\angle BAC$, and meet the bisector in F. Since $AD = AE$, AF is common, and $\angle FAD = \angle FAE$, $\triangle ADF = \triangle AEF$; $\therefore \angle AFE = \angle AFD$; *i.e.*, DE is $\perp$ to AF.

123. *If the bisectors of two angles of a triangle form by their intersection an isosceles triangle, the given triangle is isosceles.*

Let the bisectors of $\measuredangle ABC$, ACB of $\triangle ABC$, meeting in O, be equal. $OB = OC$; $\therefore \angle OCB = \angle OBC$; $\therefore \angle ACB = \angle ABC$; $\therefore AB = AC$.

124. Let BD, CE be the lines drawn to AC, AB respectively in isosceles triangle ABC, so that $AD = AE$. Then $CD = BE$ (Ax. 3), BC is common, and $\angle C = \angle B$; $\therefore \triangle DCB = \triangle EBC$; $\therefore BD = CE$.

125. Let AD, AE be lines drawn to the base of isosceles triangle ABC, so that $BD = CE$. Then $AB = AC$, $BD = CE$, and $\angle B = \angle C$; $\therefore \triangle ABD = \triangle ACE$; etc.



126. Let D, E, F, be the points such that $AD = BE = CF$. Then $DB = EC = FA$ (Ax. 3); also $\angle A = \angle B = \angle C$; $\therefore \triangle AFD = \triangle BDE = \triangle CEF$; $\therefore DF = DE = EF$; $\therefore \triangle DEF$ is equilateral.

127. This exercise is a repetition of Ex. 71, which see.

128. In the quadrilateral $ABCD$ let $AB = AD$, and $\angle ADC = \angle ABC$. Join BD. Since $AB = AD$, $\angle ADB = \angle ABD$; $\therefore \angle CDB = \angle CBD$ (Ax. 3); $\therefore CB = CD$.

129. Let $\triangle ABC$ be equilateral. Since $AB = AC$, $\angle C = \angle B$; since $AB = BC$, $\angle C = \angle A$; etc. Let $\triangle ABC$ be equiangular. Since $\angle B = \angle C$, $AC = AB$; since $\angle C = \angle A$, $AB = BC$; etc.

130. Let the diagonals AC, BD intersect in O. Since ABD, CBD are isosceles triangles on the same base BD, AC bisects BD at right angles; $\therefore \triangle AOB = \triangle AOD$ and $\triangle COB = \triangle COD$.

131. In parallelogram $ABCD$ let BE, DF be perpendiculars to the diagonal AC. Since $AD = BC$ (136), and $\angle DAF = \angle BCE$ (110), rt. $\triangle AFD =$ rt. $\triangle BEC$; $\therefore DF = BE$.

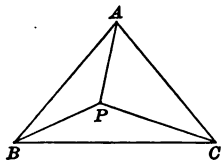
132. A square, or a rhombus, may be regarded as consisting of two equal isosceles triangles on opposite sides of the same diagonal as base; whence (74), etc.

133. Produce BP to meet AC in D. Since the exterior $\angle BPC > \angle PDC$, and exterior $\angle PDC > \angle A$ (83), $\angle BPC > \angle A$ (a. f.).

Again, since $PD + DC > PC$, $BD + DC > BP + PC$.

But $AB + AD > BD$; $\therefore AB + AC > BD + DC > BP + PC$.

134. In quadrilateral $ABCD$, let AB be the least side and CD the greatest. Join AC. Since $BC > BA$, $\angle BAC > \angle BCA$; since $CD > AD$, $\angle DAC > \angle DCA$; $\therefore \angle BAC + \angle DAC > \angle BCA + \angle DCA$; *i.e.*, $\angle BAD > \angle BCD$. In the same way, joining BD, we prove that $\angle B > \angle D$.



135. Let PA, PB, PC, be lines drawn from P to A, B, C.

Since $PB + PC < AB + AC$, and $PA + PC < AB + BC$, and $PA + PB < AC + BC$, $2(PA + PB + PC) < 2(AB + AC + BC)$;

$\therefore PA + PB + PC < AB + AC + BC$.

Again, since $PA + PB > AB$, and $PB + PC > BC$, and $PA + PC > AC$,

$$2(PA + PB + PC) > AB + AC + BC;$$

$$\therefore PA + PB + PC > \tfrac{1}{2}(AB + AC + BC).$$

136. Since AB is not less than AC, $\angle C$ is not less than $\angle B$. But exterior $\angle ADB > \angle C$; $\therefore$ $\angle ADB > \angle B$; $\therefore$ $AB > AD$.

137. Since AD, $DB = AD$, DC respectively, then (1) if $AB > AC$, $\angle ADB > \angle ADC$ (91); (2) if $AB = AC$, $\angle ADB = \angle ADC$ (69); (3) if $AB < AC$, $\angle ADB < \angle ADC$ (91).

138. Taking the preceding diagram, produce AD to E, so that $DE = AD$, and join BE. Since $AD = DE$, $BD = DC$, and $\angle BDE = \angle ADC$, $\triangle BDE = \triangle ADC$, and $BE = AC$. Then (1) $AD + DE = AE$ or $2\,AD$; $\therefore \frac{1}{2}(AB + AC) > AD$. (2) AE or $2\,AD > AB - BE$; $\therefore$ $AD > \frac{1}{2}(AB - AC)$.

139. From any point P in the bisector of $\angle BAC$, let PD be drawn $\parallel$ to AC. Since PD is $\parallel$ to AC, $\angle APD = \angle PAC = \angle PAD$; $\therefore AD = PD$.

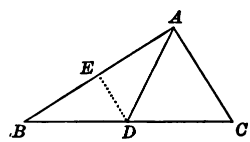
140. Since $AC = AB$, $\angle B = \angle ACB$, and since $AD = AC$, $\angle ACD = \angle D$; $\therefore$ $\angle ACB + \angle ACD = \angle B + \angle D = \frac{1}{2}$ st. $\angle =$ a right angle.

141. $\angle DBC$ is complement of $\angle C$, also $\frac{1}{2}\angle A$ is complement of $\angle C$; since $\angle A + \angle B + \angle C = \angle A + 2\angle C = 2$ rt. $\angle$s, $\angle DBC = \frac{1}{2}\angle A$.

142. See Introduction, p. 10.

143. Since the $\angle$s at P are rt. $\angle$s, $\angle C + \angle CAP = \angle B + \angle BAP$;
$\therefore$ $\angle C - \angle B = \angle BAP - \angle CAP = \angle BAM + \angle PAM - (\angle CAM - \angle PAM)$
$= 2\angle PAM$; $\therefore$ $\angle PAM = \frac{1}{2}(\angle C - \angle B)$.

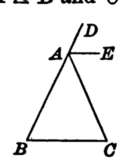
144. This interesting exercise may be solved in various ways. Thus: (1) At A, the vertex of the rt. $\angle$ in rt. $\triangle ABC$, draw AD so as to make $\angle BAD = \angle B$, and meeting BC in D. Then $DB = DA$. Now $\angle CAD$ is complement of $\angle BAD$ and $\angle C$ of $\angle B$ (120); $\therefore$ $\angle CAD = \angle C$; $\therefore$ $DC = DA = DB$; $\therefore$ DA is the median to BC and $= \frac{1}{2}BC$.



(2) Let DA be the median to the hypotenuse BC. Draw $DE \perp$ to AB; it will also be $\parallel$ to AC, which is $\perp$ to AB. Then since DE is drawn $\parallel$ to AC through the mid point of BC, E is the mid point of AB (147), and DE being $\perp$ to AB at its mid point, $DA = DB = DC$.

(3) We may also proceed by producing AD to E, so that $DE = DA$, joining EC; etc.

145. Let O be the point of meeting of the bisectors of $\angle$s B and C; draw OD, OE $\parallel$ to AC, AB respectively. $\triangle ODE$, being equiangular with $\triangle ABC$ (112), is equilateral. Now $\angle OBA = \angle OBD$ (Hyp.) $= \angle BOD$ (110); $\therefore$ $BD = DO = DE$. In the same way we show that $EC = DE$; etc.



146. Let AE be the bisector of the ext. $\angle DAC$ at the vertex of isos. $\triangle ABC$. $\angle DAC = 2\angle EAC$ (Hyp.); it also $= \angle B + \angle C = 2\angle C$; $\therefore$ $\angle EAC = \angle C$; $\therefore$ AE is $\parallel$ to BC (110).

147. See Introduction, p. 9.

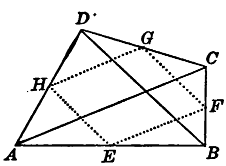
148. Let OA, OB be the bisectors of $\angle s$ A and B of the quadrilateral $ABCD$. Since $\frac{1}{2}\angle A + \frac{1}{2}\angle B + \angle O =$ a straight angle, $\angle A + \angle B + 2\angle O = 2$ straight angles. But $\angle A + \angle B + \angle C + \angle D = 2$ straight angles (125); $\therefore 2\angle O = \angle C + \angle D$; $\therefore \angle O = \frac{1}{2}(\angle C + \angle D)$.

149. See Introduction, p. 9.

150. Let PD, PE, PF, be the distances from P to AB, AC, BC, respectively, and AK the altitude from A to BC. Through P draw $GH \parallel$ to BC, meeting AB, AC, AK in G, H, L, respectively. Then PD, PE are the distances from a point P in the base, to the equal sides of $\triangle AGH$, of which AL is the altitude; $\therefore PD + PE = AL$ (Ex. 149). PK, again, being a parallelogram by construction, $PF = LK$; $\therefore PD + PE + PF = AL + LK = AK$.

151. Let D, E, F, be the mid points of the sides AB, AC, BC, of $\triangle ABC$. Since DE, DF, EF, join the mid points of the sides of $\triangle ABC$, these lines are respectively parallel to the opposite sides; $\therefore$ AF, BE, CD are par'ms; $\therefore \triangle ADE = \triangle FDE = \triangle BDF = \triangle CEF$.

152. Let AD, BE, CF, be the altitudes of $\triangle ABC$. Through A, B, C, draw GAH, GBK, KCH, $\perp$ to AD, BE, CF, respectively. Since GH, GK, HK, are $\parallel$ to BC, AC, AB, respectively (106), AK, BH, CG are parallelograms; $\therefore AH = AG$, each being equal to BC. Similarly $BG = BK$, and $CK = CH$; $\therefore AD$, BE, CF, being perpendiculars at the mid points of the sides of $\triangle KGH$, meet in a point (155).



153. Let E, F, G, H, be the mid points of the sides AB, BC, CD, DA, of the quadrilateral $ABCD$, whose diagonals are AC and BD. Join EF, FG, GH, HE. Since EF joins the mid points of AB, BC, in $\triangle ABC$, $EF = \frac{1}{2}AC$. Similarly $GH = \frac{1}{2}AC$, FG and EH each $= \frac{1}{2}BD$; $\therefore EF + GH + FG + HE = AC + BD$.

154. In $\triangle ABC$, let BE, CF be the medians to AC, AB resp., and let BE cut CF in O. Bisect BO, CO, in G, H resp., and join EF, FG, GH, HE. Since EF joins the mid points of AC and AB, EF is $\parallel$ to BC and $= \frac{1}{2}BC$. Since GH joins the mid points of OB and OC, GH is $\parallel$ to BC and $= \frac{1}{2}BC$; $\therefore EFGH$ is a par'm (142); $\therefore OE = OG = GB$, and $OF = OH = HC$; $\therefore CF$ cuts BE at two thirds of the distance from B to E. In the same way we show that the median from A to BC cuts BE at two thirds of the distance from B to E; that is, at O.

155. Let three ext. $\angle s$ of $\triangle ABC$ be ACD, BAE, CBF, formed by producing BC, CA, AB to D, E, F, respectively, and let CG, AH, BK, the bisectors of these $\angle s$, meet in G, H, K. Then $\angle GCA = \frac{1}{2}\angle ACD =$

$\frac{1}{2}(\angle A+\angle B)$; $\angle GAC=\angle EAH=\frac{1}{2}\angle BAE=\frac{1}{2}(\angle B+\angle C)$; $\therefore$ $\angle GCA + \angle GAC = \angle B + \frac{1}{2}(\angle A + \angle C)$; $\therefore$ $\angle G = \frac{1}{2}(\angle A + \angle C)$. In the same way it would be proved that the $\angle$s of $\triangle$s HAB, KBC are respectively equal to $\frac{1}{2}(\angle A + \angle B)$, $\frac{1}{2}(\angle A + \angle C)$, $\frac{1}{2}(\angle B + \angle C)$.

156. For the distance from the mid point of the common hypotenuse to the vertex of the right angle is always equal to one half of that hypotenuse (Ex. 144).

EXERCISES, pp. 79–85.

157. Suppose there could be two centers O, O', and draw a diameter through OO' to meet the circumference in A and B. Then OB, $O'B$, being each equal to OA, would be equal (Ax. 1); that is, the whole, OB, would be equal to a part, $O'B$, which is impossible (Ax. 8).

158. For the foot of the perpendicular from the center to the line will lie within or without the circle according as that perpendicular is less or greater than a radius (162).

159. Three, which is impossible (100).

160. Three, since the center, and thence the radius, would be determined by the intersection of the perpendiculars drawn at the mid points of the two lines joining the points.

161. Yes, if they have different radii; they are then said to be *concentric.*

162. See Art. 96.

163. For the perpendicular at the mid point of the one chord will pass through the center (171), and coincide with the perpendicular at the mid point of the other chord; since otherwise there could be two perpendiculars from the center to this chord (107).

164. For its vertex will be in the perpendicular to the chord at its mid point (96).

165. Let AB, CD be the chords bisecting each other in O. If O were not the center, then a perpendicular to AB at O would pass through the center (171), and also a perpendicular to CD at O; that is, these perpendiculars would coincide, which is impossible (42). Hence O must be the center, and AB, CD, diameters.

166. *A diameter passing through the mid point of a second chord is perpendicular to it.* This does not necessarily hold true when the second chord is also a diameter.

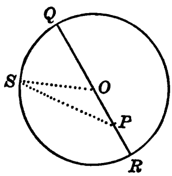
167. Let AB, AC be the equal lines; join BC. The perpendicular to BC at its mid point will bisect $\angle BAC$ (77) and pass through the center (171).

168. Draw chords subtending any two portions of the arc, and draw perpendiculars at their mid points.

169. From any point within the circle draw equal lines to the circumference, and bisect the angle thus formed.

The process may be generalized as follows: From any given point P as center, describe an arc cutting the given circumference in Q, R. Join PQ, PR; the bisector of $\angle QPR$ will pass through the center.

170. From any point A in the circumference as center, describe a circumference cutting the given circumference in B, C. From B and C as centers with the same radius, describe a third circumference, intersecting the second circumference in E and F, G and H. The lines joining EF, GH, which bisect chords AB, AC, will intersect at the center of the given circle.



171. Through the given point P draw the diameter $RPOQ$. Then PQ is the longest, and PR the shortest, line that can be drawn from P to the circumference. For draw any other line PS to the circumference, and join OS. $PO + OS = PQ$ but is greater than PS; again $PS + OP > OS$ or $OP + PR$; $\therefore$ $PS > PR$ (Ax. 5).

172. Through the given point P, and O the center, draw $PROQ$. Then PQ is the longest, and PR the shortest, line that can be drawn from P to the circumference. Draw any other line PTS, meeting the circumference in T and S; join OT, OS, and proceed as in the preceding exercise.

173. Since OAB is an isosceles triangle, the bisector of $\angle O$ will bisect chord AB; etc.

174. For the arcs being equal (Ex. 173), their chords are equal.

175. For AO, $OB = EO$, OF respectively, and $\angle AOB = EOF$; $\therefore$ $\triangle OAB = \triangle OEF$; $\therefore$ $\angle OAB = \angle OEF$; $\therefore$ EF is $\parallel$ to AB.

176. For the one circumference being partly within and partly without the other, different points of the one circumference must be at different distances from the center of the other (162).

177. Draw the diameter AE. Since chord $AB =$ chord AD (162), arc $AB =$ arc AD; $\therefore$ arc $EB =$ arc ED (Ax. 3).

178. Draw a diameter $MN \perp$ to AB; it bisects AB in M (172). On both sides of N lay off arcs NP, NQ, each equal to AM, and join PQ. $PQ = AB$ (174), and may easily be proved $\perp$ to MN, and therefore $\parallel$ to AB, by joining MP, MQ; etc.

179. Draw a diameter $\parallel$ to AB, and proceed as in Ex. 178.

QUESTIONS, p. 93.

180. Yes; see Ex. 169 or Ex. 170.

181. In order that circles containing equal chords may be equal,

the equal chords must be at equal distances from their respective centers.

182. It would pass through the center.

183. The lesser chord subtends the greater major arc.

184. No; for a line passing through any point within a circle must meet the circumference in two points.

185. Lay off equal arcs on both sides of the point, join the extremities of the arcs, and through the given point draw a line || to the chord.

186. Three.

187. No; for any point in the perpendicular at the mid point of the line joining the given points, being equidistant from these points, may be taken as the center of a circumference passing through both points.

188. When the three points are in the same straight line.

189. Yes; for the diagonals are equal and bisect each other.

190. Two only; one on each side of the line.

191. An infinite number on each side of the line, since any point in the perpendicular to the given line through the given point may be taken as the center of a circle tangent to the line.

THEOREMS, p. 93.

192. Let OO' be the central distance of the Ⓢ O* and O', of which rad. $O >$ rad. O'.

(1) If $OO' >$ rad. O + rad. O', then the extremity of rad. O, laid off from O on OO', will not meet the extremity of rad. O' laid off from O' on OO'. Hence ⊙ O' lies wholly without ⊙ O.

(2) If $OO' <$ rad. O − rad. O', or OO' + rad. $O' <$ rad. O, then the extremity of rad. O' laid off from O' on OO' produced will not reach the extremity of rad. O, and ⊙ O' will lie wholly within ⊙ O.

193. Let OO' be the central distance of Ⓢ O and O'.

(1) If $OO' =$ rad. O + rad. O', then the extremity of rad. O laid off from O on OO' will exactly coincide with the extremity of rad. O' laid off from O' on OO'. Hence the circles will be tangent externally.

(2) If $OO' =$ rad. O − rad. O', or OO' + rad. $O' =$ rad. O, then the extremity of rad. O laid off from O on OO' produced will coincide with the extremity of rad. O' laid off from O' on OO' produced. Hence ⊙ O' is tangent to ⊙ O internally.

194. Let OO' be central distance of Ⓢ O and O'. Since $OO' <$ rad. O + rad. O', ⊙ O' is not wholly without ⊙ O; since $OO' <$ rad. O − rad. O', ⊙ O' is not wholly within ⊙ O. Hence the circles intersect.

The relative positions of two circles may be illustrated by draw-

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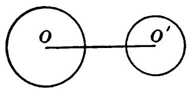
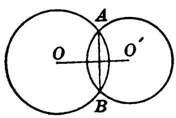
* ⊙ O signifies the circle whose center is O, whose radius is rad. O.

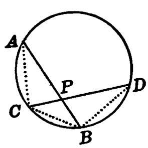
ing ⓢ O and O' on the blackboard, where $OO' >$ rad. $O +$ rad. O', and to suppose either center to move towards the other, so that in succession the central distance becomes equal to the sum of the radii, to less than their sum, to their difference, to less than their difference, to zero by the centers coinciding; and $\odot\ O'$ from being wholly external becomes tangent externally, intersecting, tangent internally, wholly within, concentric. A paper disc of the size of $\odot\ O$ may be actually moved through these various positions.



195. Let OO' be their central distance, and AB their common chord. Since O is equidistant from A and B (158), and O' is also equidistant from A and B, OO' is $\perp$ to AB at its mid point (75).



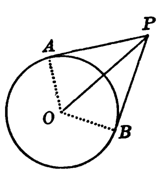
196. Let PA, PB, PC, be equal lines from P within the circumference of $\odot ABC$. Join AB, BC. Since PAB is isosceles (Hyp.), the perpendicular through the mid point of AB will pass through P (77). Similarly, the perpendicular through the mid point of BC will pass through P; hence P is the center (171).



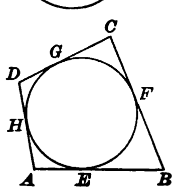
197. Let AB, CD be equal chords intersecting in P. Join AC, CB, BD. Since chd. $AB =$ chd. CD, arc $ACB =$ arc CBD; $\therefore$ arc $AC =$ arc BD (Ax. 3); $\therefore$ chd. $AC =$ chd. BD; $\therefore \triangle ACB = \triangle DBC$; $\therefore \angle ABC \doteq \angle DCB$; $\therefore PC = PB$; $\therefore PD = PA$.

198. Let AB, CD be chords intersecting radius OF in E, and making $\angle AEO = \angle DEO$. Draw OP, OQ $\perp$ to AB, CD respectively. Since hypotenuse OE is common, and $\angle OEP = \angle OEQ$, rt. $\triangle OPE =$ rt. $\triangle OQE$; $\therefore OP = OQ$; $\therefore AB = CD$.

199. Let chord AB be $\perp$ to radius OF at E. Through E draw any other chord CD, and draw $OP \perp$ to CD. Since OPE is a right angle, $\angle OEP <$ a right angle; $\therefore OE > OP$; $\therefore AB < CD$.

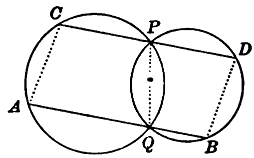


200. Let PA, PB be tangents drawn from P to $\odot\ O$. Join OA, OB. Since OAP, OBP are right angles (191), hypotenuse OP is common, and $OA = OB$, rt. $\triangle OAP =$ rt. $\triangle OBP$; $\therefore PA = PB$, and $\angle APO = \angle BPO$.

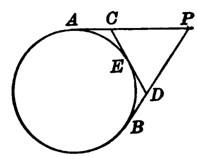


201. Let $ABCD$ be the quadrilateral, having its sides tangent to the circle in E, F, G, H, respectively. Since $AE = AH$, $BE = BF$, $CF = CG$, and $DG = DH$ (Ex. 200), $AE + BE + CG + DG = AH + DH + BF + CF$.

202. (1) Since the int. ∠s of $ABCD = 4$ rt. ∠s, and the ∠s at A and B are rt. ∠s, $\angle C + \angle D = 2$ rt. ∠s (Ax. 3); ∴ $\frac{1}{2}(\angle C + \angle D) =$ a rt. ∠, *i.e.*, $\angle CDO + DCO =$ a rt. ∠; ∴ $\angle COD =$ a rt. ∠. (2) Let CD touch the circumference in E; then $CE = CB$, and $DE = DA$; ∴ $CD = AD + BC$.



203. Let AB, CD be the parallel intercepts through P and Q respectively. Join AC, BD, PQ. Since AQ is ∥ to CP, arc $AC =$ arc PQ; ∴ $AC = PQ$. Similarly $BD = PQ$. Since in the trapezoid AP, $AC = PQ$, $\angle A = \angle PQA$, as may be shown by drawing through C a parallel to PQ. Similarly in trapezoid BP, $\angle D = \angle DPQ$. But $\angle DPQ = \angle PQA$ (110); ∴ $\angle A = \angle D$ (Ax. 1). Similarly $\angle B = \angle C$; ∴ AD is a parallelogram; ∴ $AB = CD$.

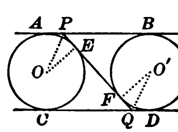


204. Let PA, PB be the tangents from P, and CD the intercepted tangent touching the circle in E. Since $CD = CE + ED = CA + DB$,

$$PC + PD + CD = PA - CA + PB - DB + CA + DB = PA + PB.$$

205. Let the circle touch AB, AC, BC, the sides of $\triangle ABC$ in D, E, F, respectively, and let P denote the perimeter. Since $P = AD + AE + BF + BD + CF + CE = 2(AD + BF + CF)$, $\frac{1}{2}P = AD + BF + CF = AD + BC$; ∴ $AD = \frac{1}{2}P - BC$.

206. Let AB, CD, the parallel sides of the trapezoid $ABCD$, touch the circle in E, G respectively, and AD, BC, the nonparallel sides, touch it in H, F respectively. $AD = AH + HD = AE + DG$ (Ex. 200) $= \frac{1}{2}(AB + DC)$.



207. Let the ⓢ O, O' touch the parallel tangents in A, B, C, D, and touch the transverse tangent PQ in E, F respectively. Join OP, OE, $O'Q$, $O'F$. $\angle PQC = \angle QPB$ (110); ∴ $\angle PQD = \angle QPA$ (44); ∴ $\angle O'QF = \angle OPE$ (Ax. 7); ∴ rt. $\triangle O'FQ =$ rt. $\triangle OEP$; ∴ $FQ = PE$, and $PE = PA$. Now $PF = PB$, and $FQ = PA$; ∴ $PQ = BA$, and $BA = OO'$, as may readily be shown by joining OA and $O'B$.

EXERCISES, pp. 97–99.

208. For if the common chord were a diameter of both, the circles would have a common center, and either be concentric or coincide.

209. Since the given chords are equal, the figure formed is a paral-

lelogram (142); and since the diagonals bisect each other, they must both be diameters (Ex. 165), and the figure is a rectangle.

210. Since O is equidistant from BQ and DQ, it is in the bisector of $\angle BQD$.

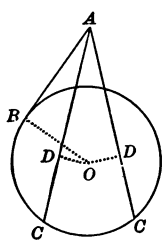
211. For if FA, DC be produced to meet in Q, OQ will be perpendicular to both AC and DF.

212. Since the int. $\angle$s of $BDOE = 4$ rt. $\angle$s, and $\angle D + \angle E = 2$ rt. $\angle$s, $\angle O + \angle B = 2$ rt. $\angle$s; *i.e.*, $\angle O$ is supplement of $\angle B$.

213. For arc $BF = \frac{1}{2}$ arc AB, and arc $BG = \frac{1}{2}$ arc BC (172, 175).

214. For $\angle B$ will be the supplement of the sum of these angles as well as of $\angle O$ (Ex. 216).

215. (1) The tangents are perpendicular to the diameter (191). (2) If the tangents are parallel to each other, the diameter perpendicular to the one must be perpendicular to the other (107). (3) The angle formed by the tangents being supplementary to that formed by the radii, that being a right angle, the tangents form a right angle (Ex. 216).



216. For of the angles of the quadrilateral thus formed, two being right angles (191), the remaining two are supplementary.

217. Shown as in Ex. 200.

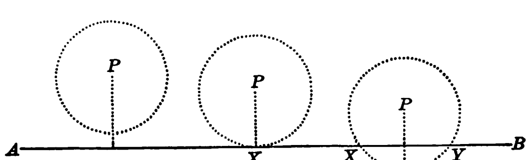
218. Let AB be the tangent and AC the secant to $\odot O$. Join OB, and draw $OD \perp$ to AC. (1) When AC lies between O and AB, since OB is $\perp$ to AB (191), and OD is $\perp$ to AC, $\angle BOD = \angle BAD$ (119), both being acute. (2) Since two of the angles of the quadrilateral $ABOD$ are right angles (Const.), $\angle BOD$ is supplementary to $\angle BAD$.

219. In the construction given in Prop. XVIII., take $A = 2B$.

220. Construct a square (206, Scholium), and draw the diagonals. As these diagonals are equal and bisect each other, their intersection is equidistant from the vertices.

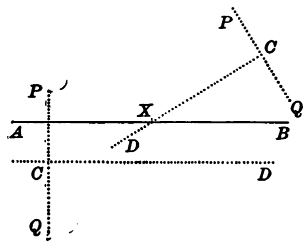
LOCI, p. 103.

221. From P as center, with CD as radius, describe a circumference. This circumference is the locus of all points situated at the

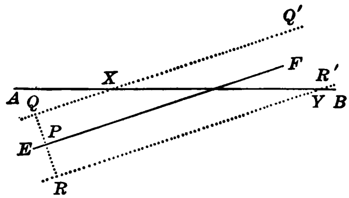


distance CD from P (212). (1) If CD is less than the ⊥ from P to AB, the circumf. will not meet AB; *i.e.*, there will be no point as required; (2) if CD is equal to the ⊥ from P to AB, the circumf. will touch AB in a point X; *i.e.*, there will be one point as required; (3) if CD is greater than the ⊥ from P to AB, the circumf. will cut AB in, say, X and Y; *i.e.*, two points can be found as required.

222. Join PQ, and at the mid point of PQ erect a ⊥ CD. CD is the locus of all points that are equidistant from P and Q (213). (1) If PQ is ⊥ to AB, CD is ∥ to AB, and cannot meet it; *i.e.*, no point can be found as required. (2) If PQ is not ⊥ to AB, CD will cut AB in some point X (114); *i.e.*, one point can be found as required.

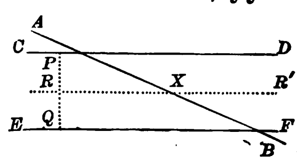


223. Through any point P in EF draw a ⊥ QPR, and make $PQ = PR = CD$. Through Q and R draw QQ', RR' ∥ to EF. QQ', RR' constitute the locus of all points situated at the distance CD from EF (214).

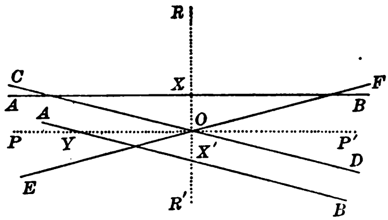


(1) If EF is ∥ to AB (case not shown in the diagram), QQ' and RR' are also ∥ to AB, and either cannot meet it, or one of them will coincide with it, if CD is equal to the distance from P to AB. Hence if EF is ∥ to AB, there is either no point as required or there is an infinite number of such points. (2) If EF is not ∥ to AB, QQ' and RR' are also not ∥ to AB, and will meet it in X and Y respectively; *i.e.*, two points can be found as required.

224. From P, any point in CD, draw PQ ⊥ to EF; it is also ⊥ to CD. Through R, the mid point of PQ, draw RR' ∥ to CD and EF.

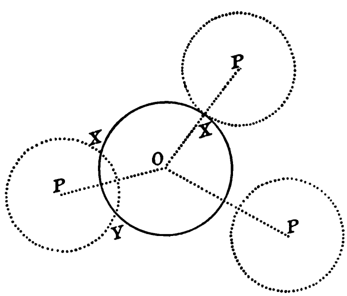


RR' is the locus of all points equidistant from CD and EF (215). (1) If CD and EF are ∥ to AB, RR' is also ∥ to AB, and can have no point in common with it unless AB happens to coincide with RR'. Hence if CD, EF are ∥ to AB, there is either no point as required or there is an infinite number of such points. (2) If CD, EF are not ∥ to AB, RR' is not ∥ to AB, and will meet it in some point X; *i.e.*, one point can be found as required.

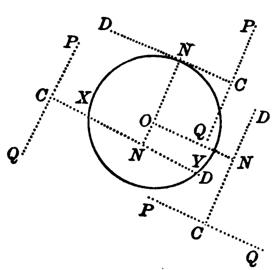


225. Let CD, EF intersect in O. Draw PP', the bisector of the opposite ∠s COE, DOF, and RR', the bisector of ∠s COF, DOE. PP' and RR', which constitute the locus of all points equidistant from CD and EF (216), are perpendicular to each other (Ex. 113). (1) If one bisector, as PP', is parallel to AB, then RR', the other, is perpendicular to AB, and therefore cuts it in some point X, and there is one point as required. (2) If neither PP' nor RR' is parallel to AB, both must cut it, as in X', Y, and there are two points as required.

226. Join P with O, the center of the given circle, and from P, with radius equal to CD, describe a circumference. This circumference is the locus of all points situated at the distance CD from P (212). (1) If $PO >$ rad. O + rad. P, then ⊙ P lies wholly without ⊙ O (Ex. 192), and there is no point as required. (2) If $PO =$ rad. $O \pm$ rad. P, then ⊙ P touches ⊙ O in a point X, and there is one point as required. (3) If $PO <$ rad. O. + rad. P, and $>$ rad. O – rad. P, then the circumference of ⊙ P intersects that of ⊙ O in two points as X, Y (Ex. 194), and there are two points as required.



227. Join PQ, and at the mid point C of PQ draw a ⊥ CD, and from O the center draw ON ⊥ to CD. CD is the locus of all points equidistant from P and Q, and ON is the shortest distance from O to CD. If $ON >$ rad. O, CD lies wholly without ⊙ O (Ex. 158), and there is no point as required. (2) If $ON =$ rad. O, CD touches ⊙ O at N, and there is one point as required. (3) If $ON <$ rad. O, CD intersects the circumference of ⊙ O in X, Y (Ex. 158), and there are two points as required.



228. Draw the parallels that consti-

tute the locus of all points situated at the distance CD from EF (214). It will not be difficult to show that, according to the distance of the center of ABM from EF, there may be no point as required, or one, or two, or three, or four.

229. Draw the line that is the locus of all points equidistant from CD and EF (215). It will not be difficult to show that, according to the distance of the center of ABM from this locus, there may be no point as required, or one, or two.

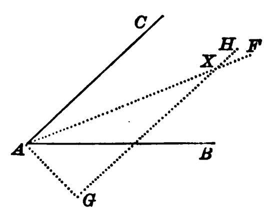
230. Draw the bisectors of the opposite angles formed by CD, EF. It will not be difficult to show that, according to the relative position of this locus and the given circle, there may be no point as required, or one, or two, or three, or four.

231. Draw the locus of points equidistant from P and Q (213), and that of points situated at the given distance from AB (214). It will be found that, if PQ is perpendicular to AB, there is no point as required; otherwise there are two.

232. Draw the locus of points equidistant from P and Q (213), and that of points equidistant from the parallels AB, CD. It will be found that, if PQ is perpendicular to AB and CD, there is no point as required; otherwise one.

233. Draw the locus CD of points equidistant from P and Q (213), and EE', FF', the locus of points equidistant from AB and CD, intersecting in O (216). It will be found that according as CD passes through O, is parallel to EE' or FF', or is parallel to neither, there is no point as required, or one point, or two.

234. Draw AF, the bisector of $\angle BAC$; at A draw $AG \perp$ to AC and equal to DE; through G draw $GH \parallel$ to AC, and cutting AF in X. Since AF is the locus of all points within $\angle BAC$ that are equidistant from AC and AB, and GH is the locus of all points within $\angle BAC$ that are at a distance equal to DE from AC, X is the required point.



235. From P as center, with radius equal to AB, describe a circumference, the locus of points situated at the distance AB from P. From Q as center, with radius equal to CD, describe a second circumference, the locus of points situated at the distance CD from Q. It may be shown, as in Ex. 226, that according as PQ is related to AB and CD, there will be no point as required, or one point, or two.

236. At A draw $AF \perp$ to AB and equal to DE; at F draw $FG \parallel$ to AB to meet AC in G. GX, the bisector of $\angle AGF$, is part of the

locus. For through P, any point in GX, draw $HPK \perp$ to AB and therefore to FG, and from P draw $PL \perp$ to AC. Then rt. $\triangle PLG =$ rt. $\triangle PHG$ (73); $\therefore PL=PH$; $\therefore PL+PK=PH+PK=AF=DE$.

237. In AC find a point G whose distance from AB is equal to DE (Ex. 223). From G draw $GF \perp$ to AB, and $GH \parallel$ to AB. The bisector GX of $\angle CGH$ is the required locus.

For through any point P in GX draw $PHK \perp$ to AB, and therefore to GH, and draw $PL \perp$ to AC. Then rt. $\triangle PHG =$ rt. $\triangle PLG$ (73); $\therefore PL = PH$; $\therefore PK - PL = PK - PH = HK = GF = DE$.

238. Let OC, OD be lines drawn from O to AB. Draw $OE \perp$ to AB, and through F, the mid point of OE, draw $XFY \parallel$ to AB. XY is the required locus. For passing through the mid point of OE, a side of $\triangle$s OEC, OED, it will also pass through the mid points of OC and OD.

239. Let AB be the wall, AC the ladder, and BC the distance between the foot of AB and that of AC. As AC is pulled away, it is always the hypotenuse of a right triangle; and its mid point, being always at a distance from B equal to one half of AC (Ex. 144), will describe a quadrant, which is the required locus.

Analysis of Problems.

The first general direction given for the solution of problems, on page 105 of the Geometry, reads as follows:

1. *Construct a diagram in accordance with the statement of the problem, as if the required construction were effected.*

In doing this, the advice to prepare the diagram in the order most convenient for its production is not merely a matter of convenience, but of necessity, seeing that the performance of the required construction is the very thing we have to discover. Hence we either construct the figure in the reverse order of the statement, or else make a figure that satisfies the requirements as nearly as possible. In preparing the diagram for Ex. 243, for example, we do not first take a line as altitude and proceed to construct the equilateral triangle, but we first construct an equilateral triangle, and draw one of its altitudes, thus providing ourselves with a diagram in which we can study out the relations of the lines, angles, etc., in the process of analysis. If, again, the problem is to describe a circle passing through two given points and touching a given line, we first describe a circle, draw a line touching it, mark two suitable points in its circumference, and thus have our diagram. In preparing the diagram for Ex. 252, however, it will be found most feasible first to construct the triangle, then to judge by

the eye, aided, perhaps, by sliding a straightedge parallel to the base, where X shall be so that XY will be equal to the intercepts between it and the base. The second and third general directions read:

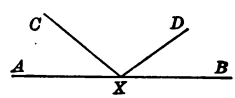
2. *Study the relations of the lines, angles, etc., in the diagram, so as to discover whether the assumed solution can be made to depend upon some known problem or theorem, especially those concerning loci.*

3. *If such dependence cannot be found by means of the original diagram, make such additions to it as the case may suggest, by joining points, drawing parallels or perpendiculars, etc., and proceed as in 2.*

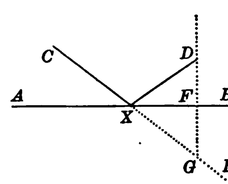
We assume the problem to be solved, the required construction to be effected, and, from the assumed relations of the lines, etc., deduce other relations, making additions to the original diagram if necessary, until we arrive at some relation that we know to be true between what is given in the diagram and what is assumed. Thus in Ex. 244, if we assume X to be the required point in the side AB of $\triangle ABC$, a moment's reflection reminds us that if X is equidistant from AC and BC, the line joining X with C is the bisector of $\angle ACB$. That is, from the assumed relation that X is equidistant from AC and BC, we have deduced a new relation which, on that assumption, is known to be true (101), and base upon it the synthesis.

Draw CX, bisecting $\angle C$ and meeting AB in X.

Since X is a point in the bisector of $\angle C$, X is equidistant from AC and BC (101).



Suppose Ex. 248 is given for solution. To prepare the diagram, in a line AB take any point X, draw equal angles CXA, DXB, and mark the points C and D, taking care not to make $XC = XD$, as that would make the diagram for a special and easy case of the problem. On studying this first diagram, we see that if CX be produced through X to E, BXE, the angle thus formed, will be equal to $\angle CXA$ (50), and therefore also to $\angle DXB$, so that XB is the bisector of $\angle DXE$. This suggests that the solution may depend upon some property of bisectors of angles. We soon see, however, that the property that the bisector is equidistant from the sides of the angle is not available here, seeing that the sides are not given, but only a point D in one of them. This suggests another property, that the bisector bisects any intercept drawn perpendicular to it (77), since such a perpendicular cuts off an isosceles triangle. We accordingly draw $DF \perp$ to AB, and produce to meet CE in G. We see



at once that $\triangle DFX = \triangle GFX$ (63), and $DF = FG$, and we thus have the relation on which to base the following synthesis.

Draw DF perpendicular to AB, and produce to G so that $FG = DF$. Draw CG, cutting AB in X. X is the point required.

Join DX. Since FX is the perpendicular at the mid point of DG, $XD = XG$; $\therefore \angle DXF = \angle GXF$ (77). But $\angle GXF = \angle CXA$ (50); $\therefore \angle DXF = \angle CXA$.

PROBLEMS, p. 107.

240. This is simply a special case of the problem in Art. 205; that is, B and C are each taken equal to $2A$.

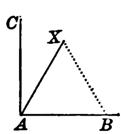
241. Let XAB be the required triangle. Since X is a right angle (Hyp.), each of the equal angles at A and B is $\frac{1}{2}$ a right angle. Hence at A erect a perpendicular to AB, and bisect the angle thus formed; etc.

242. If $ADBC$ is the required square, of which AB is a diagonal, it is evident that CAB, DAB are each a right isosceles triangle, such as that constructed in Ex. 241.

243. Let AXY be the required triangle of which AB is an altitude. If from any point in AB intercepts be drawn parallel to AX, AY respectively, the triangle thus formed is evidently equiangular, and therefore equilateral. Hence through B draw an indefinite line perpendicular to BA; on each side of B lay off equal parts BC, BD, and on CD construct an equilateral triangle CDE; then through A draw AX parallel to CE and AY parallel to ED; etc.

244. See p. 33 for analysis and synthesis.

245. Let X be the required point in the side BC of $\triangle ABC$, and suppose XD drawn parallel to AC, and XY parallel to AB. Then $ADXY$ is a square or a rhombus, according as $\angle A$ is right or is not (133, 134), and the diagonal AX bisects the angles at A and X (68, 110). Hence draw AX bisecting $\angle BAC$; etc.



246. Let AX be one of the trisectors of rt. $\angle BAC$. $\angle BAX = \frac{2}{3}$ of a right angle, which is equal to an angle of an equilateral triangle. Hence on AB, one of the sides of the right angle, construct an equilateral $\triangle XAB$; then $CAX = \frac{1}{3}$ of a right angle; etc.

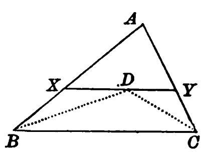
247. The analysis and synthesis of this problem are much the same as those given for Ex. 248 on p. 33.

248. This exercise and the preceding one might be enunciated together as follows: *In a given line AB find a point X such that the lines XC, XD, drawn from it to two given points C, D, shall make equal angles with AB.*

249. Let PX be the required line. It is evident that a parallel PD to AB through P will make $\angle DPX = \angle PXA = \angle C$. Hence through P draw PD parallel to AB, and at P make $\angle DPX = \angle C$; etc.

250. Let AB, CD be the given lines, and xy the required bisector. Draw $EF \perp$ to xy to meet AB, CD in E, F respectively. The $\angle$s at E and F are equal (Ex. 118, 68). Hence from any point G in AB, draw $GH \parallel$ to CD; lay off equal parts GE, GK on GA, GH resp. Join EK and produce to meet CD in F. Then $\angle GEK = \angle GKE = \angle EFD$, and the $\perp$ at the mid point of EF is the required bisector (77).

251. Let PX be the required line, cutting off equal parts OX, OP on the lines meeting in O. Since OXP is isosceles, PX will be $\perp$ to the bisector of $\angle O$. Hence draw the bisector of the angle made by the given lines, and from P draw $PX \perp$ to the bisector; etc.



252. Let XY be the intercept, in $\triangle ABC$, $\parallel$ to BC and $=$ to $BX + CY$. On XY take $XD = XB$; then $DY = YC$. Join DB, DC. Since $\triangle XBD$ is isosceles, $\angle XBD = \angle XDB$; but $\angle XDB = \angle DBC$ (110); $\therefore$ $\angle DBX = \angle DBC$; $\therefore$ BD is the bisector of $\angle ABC$. Similarly CD is the bisector of $\angle ACB$. Hence bisect $\angle$s B and C, and let the bisectors meet in D; through D draw $XDY \parallel$ to BC; etc.

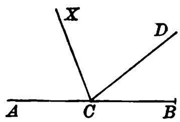
253. Let XY be the intercept in $\triangle ABC$ parallel to BC and equal to BX and CY. Join XC, YB. As BXY, CYX are equal isosceles triangles (Hyp.), $\angle XBY = \angle XYB$. But $\angle XYB = \angle YBC$ (110); $\therefore$ $XBY = YBC$; *i.e.*, BY bisects $\angle B$. Similarly CX bisects $\angle C$. Hence bisect $\angle$s B and C by BY, CX; join XY; etc.

254. Let OX, OY, OZ, be the given lines cut by XYZ so that $XY = YZ$. Draw $XP \parallel$ to OZ, cutting OY in P, and join PZ. It is easily shown that $PXOZ$ is a parallelogram, of which OP, XZ are the diagonals. Hence through any point P in OY, draw PX, $PZ \parallel$ to OZ, OX respectively, and draw XYZ; etc. It is seen that an indefinite number of lines may be drawn satisfying the given conditions.

255. Let AB be the given sum of the hypotenuse and the arm, and AYX the required triangle having the hypotenuse in AB. Join BY. Since $YA = YX$, and AYX is a rt. $\angle$, $\angle A = \frac{1}{2}$ rt. $\angle = \angle AXY$. But $\angle AXY = 2\angle B$ (122); $\therefore$ $\angle B = \frac{1}{4}$ rt. $\angle$. Hence at A make $\angle A = \frac{1}{2}$ rt. $\angle$, at B make $\angle B = \frac{1}{4}$ rt. $\angle$, and at Y, where the sides of these angles meet, make $\angle AYX =$ a rt. $\angle$; etc.

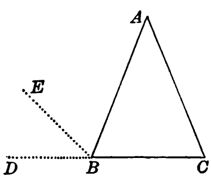
256. Let AB be the difference of the hypotenuse and the arm, and AYX the triangle having its hypotenuse in AB produced. Join BY. Since $YA = YX = XB$, and AYX is a right angle, $\angle A = \frac{1}{2}$ a right angle $= \angle X$; also $\angle YBX = \angle BYX$. Now $\angle YBX + \angle BYX + \angle X$

$=2$ right angles; $\therefore 2\angle YBX = 2$ right angles $-\frac{1}{2}$ a right angle; $\therefore \angle YBX = \frac{3}{4}$ a right angle, and $\angle ABY = \frac{5}{4}$ a right angle. Hence at A make $\angle A = \frac{1}{2}$ a right angle, at B make $\angle ABY = \frac{5}{4}$ a right angle, and at Y, where the sides meet, make $\angle AYX =$ a right angle; etc.



257. At any point C in a line AB make $\angle BCD$ equal to one of the given angles, and make $\angle DCX$ equal to the other; then ACX is the required angle.

258. Let XYZ be the required isosceles triangle, having its arms XY, XZ passing through the given points C and D respectively, its vertex X in the given line AB, and its altitude XF equal to a given line. Since the altitude XF bisects the vertical angle, $\angle AXY = \angle BXZ$ (Ax. 3). Hence the point X can be found by means of the construction of Ex. 248; then erect the perpendicular XF of the given length, and through F draw YZ parallel to AB to meet XC, XD in Y, Z respectively; etc.



259. Let ABC be the required triangle, having the given base BC, and the vertical angle equal to a given $\angle M$. This $\angle M$ is evidently the supplement of $\angle$s B and C, that is of $2\angle B$. Hence at B in CB produced, make $\angle DBE$ equal to $\angle M$, bisect $\angle EBC$ by BA, at C make $\angle C = \angle CBA$; etc.

260. Here we evidently need only to make at B and C, the extremities of the given base, angles each equal to the given angle.

261. Draw the arm AB, make at A an angle BAC equal to the given vertical angle, lay off AC equal to AB; etc.

262. Let M, N be the given sides, and O the given angle. Make an $\angle BAC$ equal to $\angle O$; on the one side lay off $AB = M$, and on the other $AC = N$; etc.

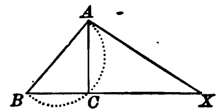
263. Let M be the given base, R and S the given base angles. Draw the base $BC = M$; at B make $\angle B = \angle R$, at C make $\angle C = \angle S$; etc.

264. Let AB be the least side. From A as center, with radius equal to $\frac{3}{2}$ AB, describe an arc CD; from B as center, with radius equal to $\frac{3}{2}$ of the preceding radius, describe an arc CE intersecting the preceding arc in C. Join CA, CB; etc.

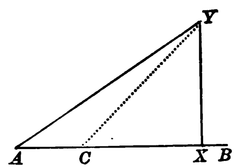
265. Let M be the given sum, N the given difference, of the hypotenuse and an arm. Find the sum, $M+N$, and the difference, $M-N$, of M and N; then $\frac{1}{2}(M+N)$ is equal to the required hypotenuse, and $\frac{1}{2}(M-N)$, to the required arm.

REMARK. — Since the hypotenuse and arm of an isos. rt. $\triangle$ have a fixed relation of magnitude, the given lines M and N must have that relation.

266. Let ABX be the triangle, AB the given arm, and AC the altitude upon the hypotenuse. ACB being a right triangle, the distance from the mid point of AB to C is equal to $\frac{1}{2} AB$ (Ex. 144). Thus a semicircle can be described upon AB as diameter, and having CA, CB as chords. Hence upon AB as diameter, describe a semicircle, place in it the chord AC equal to the given altitude, and join BC. Draw $AX \perp$ to AB to meet BC produced in X; etc.



267. Let AXY be the required triangle, AY being the hypotenuse, and AX the greater arm, exceeding the lesser arm XY by AC, or M. Join CY; then CXY is an isos. rt. $\triangle$, and $\angle XCY = \frac{1}{2}$ rt. $\angle$. Hence on AB, a line equal to the given hypotenuse, lay off $AC = M$. At C make $\angle BCY = \frac{1}{2}$ rt. $\angle$; then with A as center, and radius AB, describe an arc cutting CY in Y. Join AY, and draw $YX \perp$ to AB; etc.

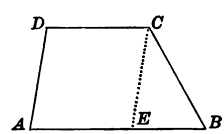


268. Let $ABCD$ be the required parallelogram, having AC as the given diagonal, AB and AD as the given sides. It is obvious that ABC is a triangle whose sides are given, since $BC = AD$. Hence we construct the $\triangle ABC$ having its sides equal to the three given lines. We then complete the parallelogram by drawing parallels to AB and BC; etc.

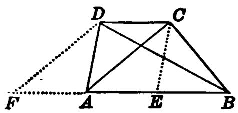
269. Let $ABCD$ be the required parallelogram, having the side AB given, and also the diagonals AC, BD. Let O be the intersection of AC, BD. It is obvious that OAB is a triangle whose sides AB, OA equal to $\frac{1}{2} AC$, and OB equal to $\frac{1}{2} BD$, are given; etc.

270. Let $ABCD$ be the required parallelogram having the diagonals AC, BD, and their included angle AOB, given. Here, again, we have a triangle of which the sides OA equal to $\frac{1}{2} AC$, OB equal to $\frac{1}{2} BD$, and their included angles are given; etc.

271. Let $ABCD$ be the required trapezoid, of which two of the sides, say AB and CD, are given as parallel. Through C draw CE parallel to AD. It is obvious that the figure consists of the parallelogram AC and the $\triangle CEB$. The $\triangle CEB$ can be constructed, since its sides CE, or AD, and BC have been given as the non-parallel sides, and $EB = AB - AE$ or CD. Hence on AB, the greater of the parallel sides, lay off AE equal to the lesser; on EB as base construct $\triangle CEB$, having its sides equal to AD and BC respectively. Through C draw CD parallel to AB and equal to AE; join DA; etc.



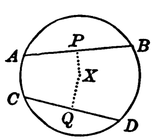
272. Let $ABCD$ be the required trapezoid, of which AB, CD, the parallel sides, and AC, BD, the diagonals, are given. Draw CE parallel to AD. ED is a parallelogram, and EC is equal to AD. If now we slide $\triangle AEC$ along BA till EC coincides with AD, and EA takes the position AF, we have a triangle DFB of which the base $FB = FA + AB = DC + AB$, and the sides DF, DB are equal to the given diagonals. Hence produce the given side BA to F so that $AF = CD$, the other given side. On BF as base construct the $\triangle DFB$, having DF, DB equal to the given diagonals. From D draw DC parallel to AB, and equal to the other given side; join DA, CB; etc.



REMARK.—This device of sliding a side or part of a figure parallel to itself will often be found useful in problems concerning quadrilaterals.

273. Let $ABCD$ (see preceding diagram) be the trapezoid, of which AB, CD, the parallel sides, BD a diagonal, and AOB, the angle formed by the diagonals, are given. Slide $\triangle AEC$ as before, and it will be seen that $\angle BDF$ of $\triangle BDF$ is equal to $\angle AOB$ of $\triangle AOB$. Now we can construct $\triangle BDF$, since we have the sides BD and BF, and the angle at D. Hence at an extremity of the given diagonal make $\angle BDF$ equal to the given angle; then from B as center, with radius $BF = AB + CD$, describe an arc cutting DF in F. On BF lay off BA, from D draw $DC \parallel$ to AB and equal to CD; join DA, CB; etc.

274. Let XPY be the required chord passing through P in $\odot O$. Join OP. OP is perpendicular to XY (177). Hence join OP, and through P draw XPY perpendicular to OP. XY is the least chord through P (Ex. 199).



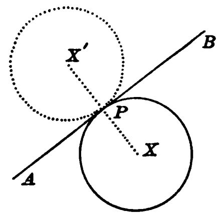
275. Let AB, CD be the chords given in position and magnitude. At P, Q, the mid points of AB, CD respectively, draw perpendiculars. These will meet in X (171). Hence, etc.

276. Let XY be the required chord equal to $2OP$, its distance from the center. Join OX. Since OP is equal and perpendicular to XP, $\angle POX = \frac{1}{2}$ a right angle. Hence draw a radius OR, and at O make $\angle ROX$ equal to $\frac{1}{2}$ a right angle; draw $XPY \perp$ to OR at P; etc.

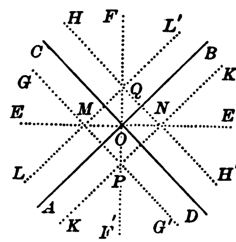
277. Let XY be the required diameter of $\odot O$, and CD the given distance of O from Z, the intersection of XY produced with AB the given line. It is evident that the problem is to find a point in AB which shall be at a given distance from O, as in Ex. 221. Hence from O as center, with radius equal to CD, describe an arc cutting AB in Z; draw ZO cutting the given circumference in Y and X; etc. The

problem will be impossible, will have one solution, or two, according to the conditions laid down in Ex. 221.

278. Let X be the center of the required circle touching AB in the given point P, and having a radius equal to CD. Join XP. XP is $\perp$ to AB (191). Hence at P draw $PX \perp$ to AB and equal to CD; etc. It is evident that since X may be on either side of AB, there are two solutions.

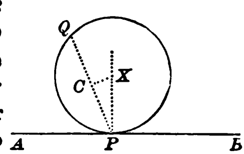


279. Let AB, CD intersect in O. Draw the bisector EE' of $\angle s\ AOC$, BOD, and the bisector FF' of $\angle s\ AOD$, BOC. EE', FF' constitute the locus of circles tangent to both AB and CD. Draw GG', HH', each $\parallel$ to CD at the given distance R, and KK', LL', each $\parallel$ to AB at the distance R. GG', HH' constitute the locus of circles with radius R tangent to CD, and KK', LL' the locus of circles with radius R tangent to AB. Hence, M, N, P, Q, the intersections of these loci, are the centers of the required circles, and there are four solutions.



280. On constructing the diagrams, it will be evident that four circles can be described as required, each having its center at the given distance from the intersection of the given lines, and in the bisector of the angle in which it lies, and having a radius equal to the perpendicular from that center to one of the lines to be touched.

281. Let $\odot X$ touch the given line AB in P and pass through Q. Join PQ. PQ being a chord of $\odot X$, the center will lie in a perpendicular at the mid point of PQ; and AB being a tangent to $\odot X$, the center will also lie in the perpendicular drawn to AB at P. Hence join PQ, and draw $CX \perp$ to PQ at its mid point C; at P draw $PX \perp$ to AB and cutting CX in X; etc.

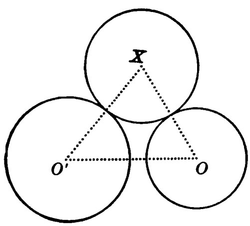


282. 1°. Let the given lines be parallel. Through P, the given point, draw $PO \parallel$ and $APB \perp$ to the given line. On PO lay off PX, PY, each $= PA$; then X, Y, are each equidistant from AB and the given lines, etc.

2°. Let the given lines meet in O. Join OP and draw $APB \perp$ to OP. Draw AX, AY, bisecting the angles at A and meeting OP in X, Y, respectively. X and Y are each equidistant from OA, OB, AB, etc.

If the perpendicular at the mid point of DE is parallel to the bisector, there is no solution; if that perpendicular coincides with the bisector, there is an infinite number of solutions, since every circumference that touches AB, AC, will pass through D and E.

283. Let $\odot P$ be the required circumference, having its center at P and bisecting the circumference of $\odot O$ in X and Y. Since P is equidistant from X and Y, the line joining PO will be perpendicular to the diameter XY. Hence join PO, and draw the diameter $XY \perp$ to PO; then from P as center, with radius PX, etc.



284. Let $\odot X$ be the required circle touching Ⓢ O and O', and having a radius equal to a given line M. Draw the central distance OO', and join OX, $O'X$. $\triangle XOO'$ has for base the central distance OO', and for sides the sum of the radii of $\odot O$ and $\odot X$, and the sum of the radii of $\odot O'$ and $\odot X$. Hence draw OO', and upon OO' as base construct a $\triangle XOO'$ with sides equal to rad. $O + M$ and rad. $O' + M$, respectively. In order that the problem may be possible, evidently we must have OO' not greater than rad. O + rad. $O' + 2M$.

QUESTIONS, p. 116.

285. $\frac{12}{7}$, $\frac{12}{13}$, $\frac{12}{18}$ or $\frac{2}{3}$, $\frac{12}{28}$ or $\frac{3}{7}$, $\frac{12}{18}$ or $\frac{2}{3}$.

286. $\dfrac{112}{3\frac{1}{2}} \div \dfrac{105}{2\frac{1}{3}} = \dfrac{224}{7} \times \dfrac{7}{315} = \dfrac{32}{45}.$

287. According to Avoirdupois weight, the ratios are: $\dfrac{16}{9}$, $\dfrac{16}{33}$, $\dfrac{1}{2\frac{1}{2}}$ or $\dfrac{2}{5}$, $\dfrac{1}{20\frac{5}{8}}$ or $\dfrac{8}{165}$. According to Troy weight, the ratios are: $\dfrac{12}{9}$ or $\dfrac{4}{3}$, $\dfrac{12}{33}$ or $\dfrac{4}{11}$, $\dfrac{2}{5}$, $\dfrac{8}{165}$.

288. $\dfrac{216}{13\frac{1}{2}} = \dfrac{432}{27} = \dfrac{16}{1}.$

289. $1°$ F. : $1°$ C. $= 100 : 180 = 5 : 9$. $1°$ F. : $1°$ R. $= 80 : 180 = 4 : 9$.

290. $\dfrac{50}{90} = \dfrac{5}{9}$. $\dfrac{50}{\frac{1}{2}(180 - 50)} = \dfrac{10}{13}.$

291. $\dfrac{75}{90} = \dfrac{5}{6}$. $\dfrac{75}{180 - 75 \times 2} = \dfrac{5}{2}.$

292. Let $\angle A$ denote a base angle; then

(1) $2\angle A + \frac{8}{5}\angle A = 180°$; $\therefore \angle A = 50°$.

(2) $\angle A : \text{rt.} \angle = 50 : 90 = 5 : 9$.

293. $\frac{35}{90-35} = \frac{35}{55} = \frac{7}{11}$.

294. Let x denote the number of degrees in the angle. Then

$$x° : 60° = 60° : 90°; \; \therefore x° = 40°.$$

EXERCISES, pp. 119–135.

295. Let $ABCD$ be the parallelogram, and AC the diagonal bisecting $\angle BCD$. Since $\angle ACD = \angle ACB$ (Hyp.), and $\angle ACD = \angle BAC$ (110), $\angle BAC = \angle ACB$; $\therefore BA = BC = CD = AD$.

296. Let $ABCD$ be the par'm. Since $AB + CD = AD + BC$ (Ex. 201), while $AB = CD$ and $AD = BC$ (136), $AB = BC = CD = DA$.

297. The two radii drawn to each point of contact are in the same straight line, being each $\perp$ to the tangent at that point. Hence each coinciding with a pair of equal radii, the central distances are equal.

298. Join D and E with F, the mid point of BC. Then $\triangle$s ADE, DBF, DFE, EFC, are equal, and $DBCE$ is equal to three of them. Hence $\triangle ABC : DBCE = 4 : 3$.

299. For $\angle A : \angle B = 60° : 90° = 2 : 3$.

300. Denoting the angles of the triangle by A, B, C, we have $\angle A + \frac{3}{7}\angle A + \frac{8}{7}\angle A = 180°$; $\therefore \angle A = 70°$, $\angle B = 30°$, $= \angle C = 80°$.

Now $\angle AOC = 180° - \frac{1}{2}(\angle A + \angle C) = 180° - 75°$;

$\therefore \angle AOC : \angle ABC = 105 : 30 = 7 : 2$.

301. (1) $\angle BOC = 180° - \frac{1}{2}(\angle B + \angle C) = 180° - 55° = 125°$;

$\therefore \angle BOC : \angle BAC = 125 : 70 = 25 : 14$.

(2) $\angle AOB = 180° - \frac{1}{2}(\angle A + \angle B) = 180° - 50° = 130°$;

$\therefore \angle AOB : \angle ACB = 130 : 80 = 13 : 8$.

302. Since arc AC = arc BD, chd. AC = chd. BD. Hence $ACBD$ is either a rectangle, or a trapezoid having its nonparallel sides equal; in either case $\angle BAC = \angle ABD$ (Ex. 88).

303. Let AD, BC intersect in X. $\triangle ACD = \triangle BDC$ (69); $\therefore \angle XCD = \angle XDC$; $\therefore \triangle XCD$ is isosceles, and a perpendicular from X will pass through the mid point of CD (77) and coincide with MN; *i.e.*, X lies in MN.

304. For that perpendicular will pass through O (171) and coincide with the diameter MN.

305. Let AB, CD, each tangent to $\odot O$ at B, D respectively, make equal angles OAB, OCD with an intercept AOC. Join OB, OD. Since $\angle$s AOB, COD are equal (44), and $OB = OD$, $\triangle AOB = \triangle COD$; etc.

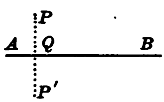
306. Proof same as in Ex. 305.

307. Let AC, BD be the diagonals of parallelogram $ABCD$ inscribed in circle O. Since $AB = CD$ (136), arc $AB =$ arc CD; and since $AD = BC$, arc $AD =$ arc BC; $\therefore$ arc $BAD =$ arc BCD; $\therefore$ BD is a diameter. Similarly AC is a diameter; $\therefore$ AC, BD intersect in O.

308. Since $OA = OB = OC$ (Ex. 307), $\angle OAB = \angle OBA$ and $\angle OAD = \angle ODA$; $\therefore$ $\angle BAD = \angle OAB + \angle ODA =$ a right angle (Ex. 100); $\therefore$ $ABCD$ is a rectangle.

309. This follows from Ex. 308, but may be proved independently from the consideration that $OA = OB = OC = OD$.

310. Draw a perpendicular at the mid point of the line joining the given points. Its intersection with the given line will be the required center; etc. If the line joining the given points is perpendicular to the given line, there will be no solution.



311. Let P be the given point, and AB the given line. From P draw PQ perpendicular to AB, and produce to P', so that $QP' = QP$. Since P and P' are equally distant from every point in AB (213), any circumference that has its center in AB, and passes through P, must also pass through P' (212).

312. Join P, the point, with O, the center of the given circle, cutting the circumference in X and again in Y. From P as center, with radii PX, PY, describe circles. Both will satisfy the given conditions.

313. For if a radius be drawn cutting off from the greater angle an angle equal to the less, the arc subtended by this part will be less than the arc subtended by the whole angle.

314. For two diameters intersecting at right angles divide the circumference into four equal arcs (257), or quadrants; *i.e.*, a right angle has a quadrant for arc; an acute angle, an arc less than a quadrant; etc.

315. For the vertical angles formed by the intersecting diameters being equal, the intercepted arcs are equal.

316. Let $ACBD$ be the figure formed by joining the extremities of the diameters AB and CD, O being the center. It is easily seen that $\angle OCA = \angle OAC = \angle OBD = \angle ODB$, and $\angle OAD = \angle ODA = \angle OCB = \angle OBC$. Hence $\angle ACB = \angle CBD$; etc. By making use of Art. 266, the proof is, of course, easier.

317. $\frac{6}{5}$ of a right angle $= 90° \times \frac{6}{5} = 108°$; $\therefore$ arc ACB : arc $ADB = 108° : (360° - 108°) = 3 : 7$.

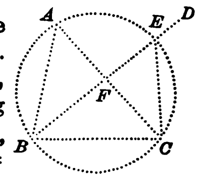
318. Since $\angle AOE = 180°$, and $\angle AOB = 108°$, $\angle BOE = 72°$; $\therefore$ arc $BE = \frac{72}{360} = \frac{1}{5}$ a circumference.

319. Find another point O' equidistant from A and B (78); then OO' bisects $\angle AOB$ and also arc ACB.

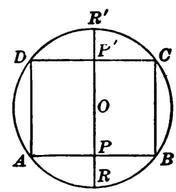
320. Since B is an angle of 32°, arc AC is an arc of 64° (264).

321. Arc $CD = 360° - (\text{arc } AC + \text{arc } AD) = 360° - 2 \text{ arc } AC$; $\therefore$ arc $AC = \frac{1}{2}(360° - 102°) = 129°$; $\therefore \angle BAC = 64° \, 30'$.

322. Let x denote the number of degrees in arc BD. Then since $\frac{1}{2}(30° + x°) = 25°$, $x = 20°$.



323. Let A, B, C, be three points known to be points of a circumference. Join AB, BC, CA. At B draw BD making any angle ABD with AB, and cutting AC in F. At C draw CE making $\angle ACE = \angle ABE$, and meeting BD in E. In $\triangle$ AFB, CFE, since $\angle ABE = \angle ACE$, and $\angle AFB = \angle CFE$, $\angle A = \angle E$ (121); $\therefore$ E is a point on the same circumference as A, B, and C, for otherwise C being joined with the point where BD does cut the circumference, there would be an exterior equal to an interior remote angle (266).



324. In $\odot$ O let OPR, $OP'R'$ be radii drawn to AB, CD, opposite sides of an inscribed parallelogram. Since $AB = CD$, $OP = OP'$ (182); $\therefore$ $OR - OP = OR' - OP'$; *i.e.*, $PR = P'R'$; $\therefore$ $OP : PR = OP' : P'R'$.

325. OA and $O'B$ will be parallel, being both perpendicular to AB (191).

326. Join PO cutting AB in C. It is easily shown that rt. $\triangle$ PAO, ACO are equiangular, so that $\angle CAO = \angle APO = \frac{1}{2} \angle P$ (Ex. 200).

327. Let x denote the number of degrees in arc BC. Then

$$17° = \tfrac{1}{2}(x° - 36°); \therefore x = 70.°$$

328. Let $n = 6$, and A, B, C, D, E, F, be the angles of the polygon.

$\angle A$ is measured by $\frac{1}{2}$ a circumference $-$ (arc AB + arc AF);

$\angle C$ is measured by $\frac{1}{2}$ a circumference $-$ (arc BC + arc CD);

$\angle E$ is measured by $\frac{1}{2}$ a circumference $-$ (arc ED + arc EF);

$\therefore \angle A + \angle C + \angle E$ is measured by $\frac{3}{2}$ a circumference $- (AB + BC + CD + ED + EF + AF)$; $\therefore \angle A + \angle C + \angle E$ is measured by $\frac{1}{2}$ a circumference; etc.

329. The problem is the same as to find the center of a circumference passing through three given points. It is impossible if the points are in the same straight line.

330. Join the points so as to form a quadrilateral. If the opposite angles of this figure are supplementary, the point required can be found (see Ex. 398).

331. The sides of the triangle being chords of the circumference that can be passed through its vertices, the perpendiculars at the mid points of these chords intersect at the center (171).

332. The greater angle is measured by one half of the greater arc.

333. The line of centers, being perpendicular to the common chord (Ex. 195), bisects both pairs of arcs subtended by that chord (175).

QUESTIONS, p. 136.

334. $\angle A = 90° \times \frac{2}{3} = 60°$; $\therefore \angle A : \angle B = 60° : 50° = 6 : 5$.

335. Let A be the vertical angle. (1) Since $\angle A = 2\angle B = \angle B + \angle C$, $\angle A + \angle A = 180°$; $\therefore \angle A = 90°$, and is measured by a quadrant. (2) Since $\angle A = \frac{1}{2}\angle B = \frac{1}{4}(\angle B + \angle C)$, $\angle A + 4\angle A = 180°$; $\therefore \angle A = 36°$, and is measured by $\frac{36}{90}$ or $\frac{2}{5}$ of a quadrant. (3) Since $\angle A = \frac{1}{n}\angle B = \frac{1}{2n}(\angle B + \angle C)$, $\angle A + 2n\angle A = 180°$; $\therefore \angle A = \frac{180°}{2n+1}$, and is measured by $\frac{2}{2n+1}$ of a quadrant.

336. $\angle A = 360° \times \frac{1}{5} = 72°$, $\angle B = 360° \times \frac{1}{8} = 45°$; $\therefore \angle C = 63°$.

337. Let A and B be the acute angles. Since $\angle A = 360° \times \frac{3}{20} = 54°$, $\angle B = 90° - 54° = 36°$; $\therefore \angle B : \angle A = 36 : 54 = 2 : 3$.

338. $360° - 230° = 130°$; $\therefore \angle C = 65° = \angle D = \angle E$.

339. $\angle ABE = 30° \times \frac{1}{2} = 15°$. $\angle BAE = 180° - (\angle E + \angle ABE)$
$= 180° - 80° = 100°$.

340. If $\angle BAC = 65°$, arc $AC = 130°$; also arc $AD = 130°$; $\therefore$ arc $CD = 100°$.

341. Arc CA + arc AD + arc $CD = 2$ arc $CA + \frac{3}{2}$ arc $CA = 360°$; $\therefore$ arc $CA = 102\frac{6}{7}°$; $\therefore \angle BAC = 51\frac{3}{7}° = 51° \, 25' \, 42.85''$.

342. Arc AC + arc $AB = 40° \times 2 = 80°$; $\therefore$ arc AD + arc $BC = 360° - 80° = 280°$.

343. Let x denote the number of degrees in arc BC. Since $x° - \frac{1}{2}x° = 25° \times 2$, $x = 100°$, and arc BC + arc $DE = 150°$; $\therefore$ arc BE + arc $CD = 210°$.

344. Since $DA = DB$, $\angle B = \angle A = n°$; $\therefore$ arc $DE = 2n°$. But arc BC − arc $DE = 2n°$; $\therefore$ arc $BC = 4n° = 2$ arc DE.

345. Arc BC − arc $BD = 3$ arc BD − arc $BD = 2$ arc $BD = 60°$; $\therefore$ arc $BD = 30°$; $\therefore$ arc $BC = 90°$; $\therefore$ arc $CD = 360° - 120° = 240°$, and $\angle B = 45°$.

346. Let n be the number of degrees in $\angle A$; then $\angle C = n°$; $\therefore$ arc $BD = 2n°$. Now arc BC − arc $BD = 2n°$; $\therefore$ arc $BC = 4n°$; $\therefore BC : BD = 2 : 1$.

347. If the angle formed by the tangents is 30°, the angle formed by the radii to the points of contact is $180° - 30° = 150°$; $\therefore$ the major arc $= 360° - 150° = 210°$.

348. $\frac{1}{2}(180° - 60°) = 60°$.

349. $360^\circ \times \frac{1}{3} - 360^\circ \times \frac{1}{4} = 30^\circ = \frac{30}{360}$ circumf. $= \frac{1}{12}$ circumf.

350. The vertical angle being 54°, each base angle $= \frac{1}{2}(180^\circ - 54^\circ) = 63^\circ$. Hence the ratio of the arcs is $54 \times 2 : 63 \times 2 = 6 : 7$.

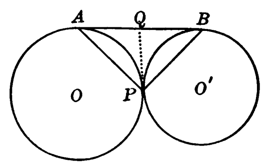
351. The vertical angle being $37^\circ\ 15'\ 32''$, each base angle $= \frac{1}{2}(180^\circ - 37^\circ\ 15'\ 32'') = 71^\circ\ 22'\ 14''$, and the ratio is $134132'' : 256934'' = 67066 : 128467$.

EXERCISES, pp. 139–154.

352. The bisector of the vertical angle divides the triangle into two equal right triangles with the right angles at the mid point of the base (77); hence the mid point of each arm is equidistant from the extremities of the arm and the mid point of the base (Ex. 144).

353. The altitude upon the third side divides the triangle into two right triangles having their common vertex in the third side, or the third side produced; and the mid point of each side is equidistant from its own extremities and that vertex (Ex. 144).

354. For the common chord, as it does not pass through the center, divides the intersected circumference unequally, the exterior segment being the greater, since it includes the center. Hence the angle in this segment is acute.

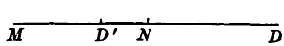


355. At P draw the common tangent PQ meeting AB in Q. Since $QA = QP$, and $QB = QP$ (Ex. 200), $\angle PAQ = \angle APQ$, and $\angle PBQ = \angle BPQ$; $\therefore\ \angle PAB + \angle PBA = \angle APB$; $\therefore\ APB$ is a right angle (Ex. 100).

356. For the radius to the point of contact is perpendicular to the tangent, and also to the chord.

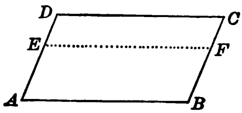
357. For $BD' : D'C = AB : AC = BD : DC$ (278, 280).

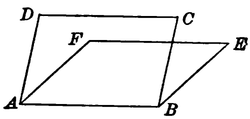
358. Since $BD' : D'C = BD : DC$ (Ex. 357), (1) BC is divided internally in D' so that $D'B : D'C = DB : DC$; (2) BD is divided into three segments, BD', $D'C$, and CD, so that BD, the whole line, is to DC, one of the outer segments, as BD', the other outer segment, is to $D'C$, the inner segment.



359. Let MN be divided internally in D', and externally in D, so that $D'M : D'N = DM : DN$; then, also, $D'N : DN = D'M : DM$ (244); that is, $D'D$ is divided internally in N and externally in M, so as to be divided harmonically.

360. For the whole line DM is to one of the outer segments DN as the other outer segment $D'M$ is to the inner segment $D'N$.





361. In any parallelogram AC, draw EF parallel to AB. Then AC and AF are mutually equiangular yet not similar, since the homologous sides are not proportional. Again, we may have two parallelograms, AC, AE, such that the determining sides DA, AB are respectively equal to FA AB; yet AC, AE are not similar, not being mutually equiangular.

362. Since $BC : B'C' = 16 : 10 = 8 : 5$, $p : p' = 8 : 5$.

363. For $AL : AD = AG : AB = GH : BC$ (283).

364. $\triangle A'AD$ is similar to $\triangle OAB$ and $\triangle OAB$ to $\triangle OA'B'$.

365. OB, OB' are homologous sides of similar $\triangle$s OAB, $OA'B'$; hence perimeter OAB : perimeter $OA'B' = OB : OB'$ (296).

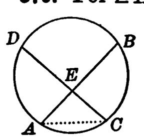
366. $\angle B + \angle C = \angle DAC + \angle DAB = \frac{3}{5} \angle DAB + \angle DAB = 90°$; $\therefore \angle DAB = 56° 15'$, $\angle DAC = 33° 45'$.

367. $BD : AD = AD : DC$; $\therefore 10 - 2 : AD = AD : 2$; $\therefore AD^2 = 16$, $AD = 4$.

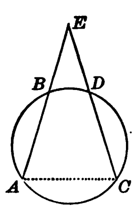
368. $BC : AB = AB : BD$; $\therefore AB^2 = 10 \times 8$; $\therefore AB = 4\sqrt{5}$. Similarly $AC^2 = 10 \times 2$; $\therefore AC = 2\sqrt{5}$.

369. Prove as in Art. 301.

370. For $\angle A = \angle D$ and $\angle B = \angle C$ (266).



371. Let AB, CD be equal chords intersecting in E. Join AC. Since $AB = CD$, arc ACB = arc CAD; $\therefore$ arc BC = arc AD (Ax. 3); $\therefore \angle A = \angle C$; $\therefore EC = EA$; $\therefore EB = ED$ (Ax. 3).



372. Let the equal chords AB, CD meet in E when produced. Join AC. Since $AB = CD$, arc AB = arc CD; $\therefore$ arc ABD = arc CDB (Ax. 2); $\therefore \angle C = \angle A$; $\therefore AE = CE$; $\therefore BE = DE$ (Ax. 3).

373. The chords being equally distant from the center are equal; $\therefore$ their segments are equal (Ex. 371); $\therefore OA : OB = OC : OD$.

374. If OA, OD are equally distant from the center, then $AB = CD$; then $OA = OD$ and $OC = OB$ (Ex. 372); $\therefore OA : OB = OD : OC$.

375. Let PT, PT' be tangents drawn from any point P in the production of AB, the common chord of the intersecting circles. Then $PT = PT'$, each being a mean proportional between PA and PB (303).

QUESTIONS, p. 155.

376. $AB : AD = AC : AE$; or $3 + 7 : 7 = 55 : AE$; $\therefore\ AE = 38.5$.

377. $\sqrt{11} + 7 : 7 = 55 : AE$; $\therefore\ AE = 37.32$.

378. $AB : AC = BD : DC$; $\therefore\ AB + AC : AB = BD + DC : BD$; $\therefore\ 9 + 7 : 9 = 12 : BD$; $\therefore\ BD = 6.75$, $DC = 5.25$.

379. $AB : AC = BD : DC$; $\therefore\ 9 : 7 = 20 : DC$; $\therefore\ DC = 15\frac{5}{9}$, $BC = 4\frac{4}{9}$.

380. If ABC is isosceles, the bisector of the exterior vertical angle is parallel to the base (Ex. 146), and cannot meet it. The proposition may then be said to hold in this way, that, the external segments being infinitely great lines, their ratio is indefinitely near to a ratio of equality, which is that of AB to AC.

381. They are similar if their third angles are both either greater or less than a right angle, or if one of them is a right angle.

382. $AB : AC = A'B' : A'C'$, or $a : b = c : A'C'$; $\therefore\ A'C' = \dfrac{bc}{a}$.

383. $AD : DE = A'D' : D'E'$, or, $4 : 3 = 3.2 : D'E'$; $\therefore\ D'E' = 2.4$.
$AD : AC = A'D' : A'C'$, or, $4 : 5 = 3.2 : A'C'$; $\therefore\ A'C' = 4$.

384. $AB^2 = BC \cdot BD$; $\therefore\ BD = AB^2 \div BC = 3.2$.
$AC^2 = BC \cdot DC$; $\therefore\ DC = AC^2 \div BC = 1.8$.
$AD^2 = BD \cdot DC, = \frac{16}{5} \times \frac{9}{5}$; $\therefore\ AD = \frac{12}{5} = 2.4$.

385. $OA : OD = OC : OB$; *i.e.*, $\frac{5}{7} OD : OD = OC : 8$; $\therefore\ OC = 5\frac{5}{7}$.

386. $CG : AG = BF : AF = BD : AD + 2\,BD = BD : mBD + 2\,BD$; $\therefore\ CG : AG = 1 : m + 2$; $\therefore\ CG = \dfrac{1}{m+2} AG$.

387. $AD : EC = AB : BE = AB : AB + AC = 10 : 10 + 7 = 10 : 17$.

388. If $\triangle ABC$ is isosceles, then $\angle BCE$ is a right angle (Ex. 140); *i.e.*, $\angle BCE = 90°$.

389. $BD : BC = AD : AE = 3 : 2$; $\therefore\ BD : BD - BC = 3 : 3 - 2$, that is, $BD : DC = 3 : 1 = AB : AC$.

390. Since CE would be parallel to AD (Ex. 146), E would coincide with B.

391. $AB + BC : BC = AD + EC : EC$; $\therefore\ AD + EC : AC = EC : BC$. Now $EC : BC = EC'$ or $BB' : OB = OB - OB' : OB = 8 - 5 : 8$; $\therefore\ AD + EC = 3\,AC \div 8 = 4.5$.

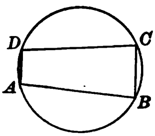
392. $OA' : OC' = OA : OC = AB : BC = 10 : 8 = 5 : 4$.

393. $AB : AC = A'B' : A'C'$; $\therefore\ AB - AC : AB = A'B' - A'C' : A'B'$; $\therefore\ AB - AC : A'B' - A'C' = AB : A'B' = BC : B'C' = m : n$.

394. $AC : A'C' = AB : A'B' = 18 : 12 = 3 : 2$.

395. $AB + DA + BD : AC + DC + DA = BD : DA = m : n$.

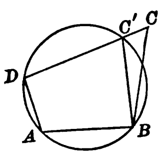
THEOREMS, p. 160.

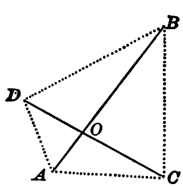


396. Let AD, BC join the extremities of the equal chords AB, CD. Since $AB = CD$, arc AB = arc CD; $\therefore$ arc ABC = arc BCD; $\therefore \angle D = \angle A$. Similarly $\angle C = \angle B$; $\therefore \angle A + \angle B = \angle C + \angle D$; $\therefore \angle A + \angle B$ = a straight angle; $\therefore$ AD is $\parallel$ to CB (113).

397. Let $ABCD$ be the quadrilateral. $\angle A$ is measured by $\frac{1}{2}$ arc BCD, and $\angle C$ by $\frac{1}{2}$ arc BAD; $\therefore \angle A + \angle C$ is measured by $\frac{1}{2}$ circumference $ABCD$; $\therefore \angle A + \angle C$ is equal to a straight angle.

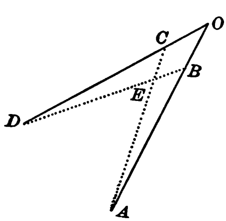
398. Let $ABCD$ be a quadrilateral such that $\angle A + \angle C$ = a straight angle. Describe a circumference through D, A, and B (185); it will also pass through the fourth vertex C. For if C should fall without the circumference, so that the latter cut CD in C', join BC'. Then as $\angle A$ is supplementary to $\angle BC'D$ (Ex. 397), and is also supplementary to $\angle C$ (Hyp.), int. $\angle C'$ = ext. $\angle C$ (44), which is impossible (83). Hence C cannot lie without circumference DAB; and, in the same way, it may be shown that C cannot lie within circumference DAB. Hence C coincides with C', and is concyclic with A, B, and D.





399. Join AC, CB, BD, DA. Since $OA : OC = OD : OB$, $\triangle\triangle$ AOC, BOD are similar (290); $\therefore \angle OBD = \angle OCA$ and $\angle OAC = \angle ODB$. In the same way we show that $\angle OBC = \angle ODA$ and $\angle OAD = \angle OCB$; $\therefore \angle OBD + \angle OBC + \angle OAC + \angle OAD = \angle OCA + \angle OCB + \angle ODB + \angle ODA$; *i.e.*, $\angle\angle$ CAD and CBD are supplementary; $\therefore$ A, B, C, D, are concyclic (Ex. 398).

REMARK.—In preparing the diagram for this exercise, describe an erasible circumference, and draw in it any chords AB, CD. Proceed in a similar manner in the preparation of the next diagram.



400. Join AC, BD, intersecting in E. Since $OA : OD = OC : OB$, $\triangle\triangle$ OAC, ODB are similar (290); $\angle OAC = \angle ODB$; $\therefore$ $\triangle\triangle$ AEB, DEC are similar (287); $\therefore AE : DE = EB : EC$; $\therefore$ A, B, C, D, are concyclic (Ex. 399).

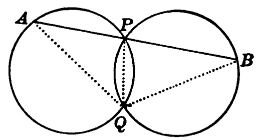
401. Let AB, AC, AD, be the chords; then $\angle BAC = \angle CAD$ (264).

402. Let AB be the tangent, AC the chord, D the mid point of

arc ADC, DE and DF perpendiculars from D to AB, AC respectively. Join AD. Since $\angle CAD$ is measured by $\frac{1}{2}$ arc DC or AD, and $\angle BAD$ is measured by $\frac{1}{2}$ arc AD, $\angle DAC = \angle DAE$; $\therefore$ rt. $\triangle DAF =$ rt. $\triangle DAE$; $\therefore DF = DE$.

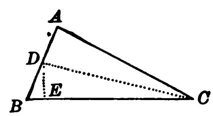
REMARK. — A, E, D, F, are concyclic.

403. Let AB, CD be the parallel sides of trapezoid $ABCD$, and AC, BD the diagonals intersecting in O. Since AB is $\parallel$ to CD, $\angle OAB = \angle OCD$; $\therefore$ $\triangle$s AOB, COD are similar; $\therefore OA : OC = OB : OD$.



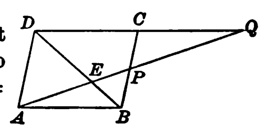
404. Draw the common chord PQ, and join QA, QB. Since PQ is common to the equal circles, the arcs it subtends are equal; $\therefore \angle A = \angle B$; $\therefore QA = QB$.

405. In rt. $\triangle ABC$, let the hypotenuse BC be divided by the altitude AD, so that $BC : BD = BD : DC$. Now, also, $BC : AC = AC : DC$ (297); $\therefore AC = BD$.

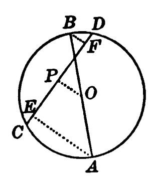


406. In $\triangle ABC$, right angled at A, let the bisector of $\angle ACB$ meet AB in D; draw DE $\perp$ to BC. Since $\angle ACD = \angle ECD$, rt. $\triangle CAD$ = rt. $\triangle CED$; $\therefore AD = DE$ and $AC = EC$; $\therefore BE = BC - AC$. Since $\angle B$ is common to rt. $\triangle$s ABC, EBD, $AB : AC = BE : DE = BC - AC : AD$.

407. In $\triangle ABC$, let CD, AE be the altitudes to any two sides AB, BC respectively. In rt. $\triangle$s CDB, AEB, $\angle B$ is common; $\therefore$ they are similar; $\therefore CD : AE \doteq CB : AB$.



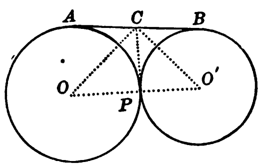
408. By similar triangles we show that $AE : EP = AD : BP = DE : EB$. Also $QE : AE = DE : EB$; $\therefore QE : AE = AE : EP$.



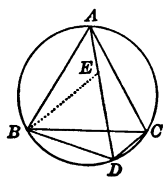
409. In circle O, let AB be the diameter and CD the chord. From A, O, and B, draw perpendiculars AE, OP, BF, to CD. Since AE, OP, and BF are parallel (106), $OA : PE = OB : PF$; $\therefore OA : OB = PE : PF$. But $OA = OB$; $\therefore PE = PF$; $\therefore O$ is equidistant from E and F (96).

410. If the triangle is isosceles, there can be only one such triangle on the chord of the segment as base (69); but there can be two scalene triangles on that chord as base, having their homologous angles at opposite extremities of the base; a third triangle, if equal to each of them, would necessarily coincide with one or other of those two (63).

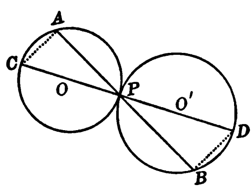
411. Join OO', draw $PC \perp$ to OO', and join OC, $O'C$. OC and $O'C$ bisect the adjacent supplementary $\measuredangle ACP$, BCP (Ex. 200); $\therefore OCO'$ is a right angle (Ex. 113); $\therefore OP : PC = PC : PO'$; $\therefore 2\,OP : 2\,PC = 2\,PC : 2\,PO'$. Now $AC + CB$ or $AB = 2\,PC$ (Ex. 200); $\therefore AB$ is a mean proportional between the diameters.



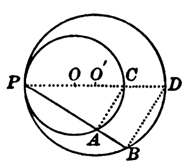
412. Let ABC be the inscribed equilateral triangle, and AD, BD, DC, the chords. On DA lay off $DE = DB$, and join BE. $\angle ADB = \angle ACB$ (266) $= \frac{1}{3}$ st. $\angle$; $\therefore$ the equal $\measuredangle DBE$, DEB each $= \frac{1}{3}$ st. $\angle$; $\therefore \triangle DBE$ is equilateral; $\therefore BE = BD = DE$. Now $\angle DBE = \angle ABC$; $\therefore \angle DBC = \angle ABE$ (Ax. 3); $\therefore \triangle DBC = \triangle EBA$ (66); $\therefore DC = AE$; $\therefore DB + DC = DE + AE = AD$.



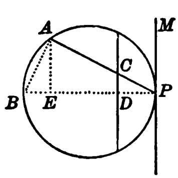
413. Let APB be a line through P, the point of contact. Produce the central distance OO' both ways, to C and D, and join AC, BD. CD passes through P (199), and APC, BPD are similar right triangles (267, 50); $\therefore AP : BP = PC : PD$, APB being any intercept through P.

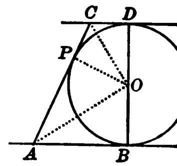


414. Let PAB be any chord of the external circle O', drawn from P to meet the circumferences in A and B respectively. Produce the central distance OO' both ways to meet the circumferences in C and D, and join AC, BD. CD passes through P (199), and PAC, PBD are similar right triangles (267); $\therefore AP : AB = PC : CD$.

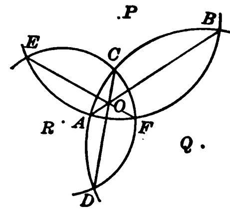


415. Let PCA be any chord drawn from P, the point of contact of tangent PM, and CD any chord $\parallel$ to PM, cutting PA in C, and PB, the diameter drawn from P, in D. Draw $AE \perp$ to PB, and join AB. Then (297) $PA : PB = PE : PA = PD : PC$. Similarly any other chord $PC'A'$ would be divided so that $PA' : PB = PD : PC'$; $\therefore PA : PC = PA' : PC'$.





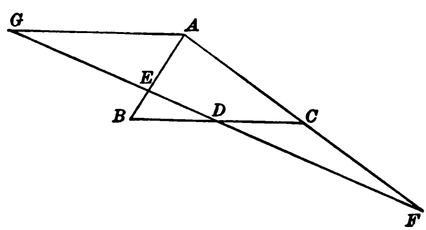
416. Let AB, CD be parallel tangents at the extremities of diameter BOD, and AC a tangent intercepted between AB and CD, and touching $\odot\ O$ in P. Draw OA, OC, OP. OA and OC bisect $\angle s\ A$ and C (Ex. 200), hence they bisect their supplementary $\angle s\ POB$, POD; $\therefore \angle AOC$ is a right angle, and OP is $\perp$ to AC; $\therefore PA : PO = PO : PC$.



417. Let P, Q, R, be the centers, so that $\odot\ P$ intersects $\odot\ Q$ in A and B, and $\odot\ R$ in E and F, while ⓢ Q and R intersect in C and D.

Let the chords AB, CD intersect in O. Since AB, CD are chords in $\odot\ Q$, $OA \cdot OB = OC \cdot OD$ (301). Join EO, and produce to meet circumf. P in X, and circumf. R in Y. Then $EO \cdot OX = OA \cdot OB$, and $EO \cdot OY = OC \cdot OD$; $\therefore EO \cdot OX = EO \cdot OY$; $\therefore OX = OY$; *i.e.*, X and Y coincide with F, the intersection of $\odot\ P$ and $\odot\ R$.

418. Let DF, drawn through the mid point of BC in $\triangle\ ABC$, meet AB in E, AC produced in F, and AG parallel to BC in G. By similar



triangles $FC : FA = FD : FG = DC : AG$; and $EB : EA = ED : EG = DB : AG = DC : AG$, since $DC = DB$ (Hyp.); $\therefore FD : FG = ED : EG$; that is, the whole line FG is to an outer segment FD as EG, the other outer segment, is to ED, the inner segment.

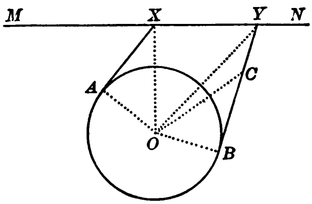
PROBLEMS, p. 162.

419. Draw that diameter of the given circle which is parallel to the given line; each of the tangents at the extremities of that diameter will be perpendicular to the given line (106).

420. Draw that diameter of the given circle which is perpendicular

to the given line; each of the tangents at the extremities of that diameter will be parallel to the given line (108).

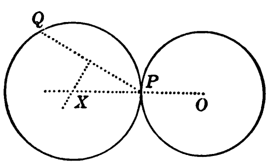
421. Let PA, PB be tangents forming $\angle P$ equal to the given angle. Join OA, OB. It is obvious that $\angle O$ is the supplement of $\angle P$. Hence at O make $\angle AOB$ equal to the supplement of the given angle; etc.



422. Let X be the required point, and Y any other point, in MN. Draw XA, YB, tangents to $\odot\, O$, and join OX, OY, OA, OB. Then since the angle formed by the tangents drawn from X is greater than the angle formed by the tangents drawn from Y (only one tangent of each pair being shown in the diagram), $\angle AXO > \angle BYO$ (Ex. 200, Ax. 7); $\therefore$ $\angle AOX$, the complement of $\angle X$, is less than $\angle BOY$, the complement of $\angle BYO$ (Ax. 5). At O draw OC making $\angle BOC$ equal to $\angle AOX$, and meeting BY in C. $\triangle BOC = \triangle AOX$ (63); $\therefore$ $OC = OX$. But $OC < OY$ (99); $\therefore$ $OX < OY$ or any other line drawn from O to MN; $\therefore$ OX is $\perp$ to MN (93). Hence draw $OX \perp$ to MN; etc.

423. The locus of points from which tangents of a given length can be drawn to a given circle, is evidently a circumference concentric with the given circle, and with radius equal to the hypotenuse of a right triangle whose arms are the given length of the tangent and the radius of the circle, respectively. Hence at the extremity of any radius draw a tangent equal to the given length; etc.

424. Let OX be the central distance of the given and required circles, cutting the given circumference in P; then PX is the radius of the required circle.

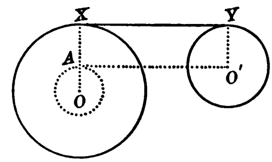


425. Let $\odot\, X$ touch the given $\odot\, O$ in P and pass through Q. Join OP and PQ. At the mid point of PQ draw a perpendicular meeting OP produced in X; etc.

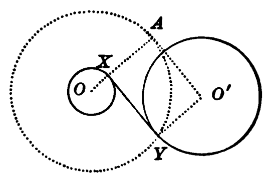
426. Let AB be the common chord, and R, R' the given radii. Upon opposite sides of AB construct isosceles triangles having their arms equal to R, R' respectively. The vertices will be the required centers.

427. Let R, R' be the radii of the given circles O and O', and R'' that of the required circle. Join OO'. From O as center, with radius equal to $R + R''$, describe an arc AC, and from O' as center, with radius equal to $R' + R''$, describe an arc BC cutting arc AC in C. C is the center of the required circle.

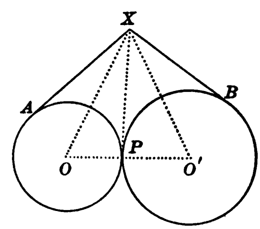
428. Let XY be an exterior tangent common to Ⓢ O and O'. Join OX, $O'Y$, and draw $O'A \perp$ to OX, meeting it in A. AY is obviously a rectangle, and $AX = O'Y$; $\therefore$ $OA = OX - O'Y$, and, being $\perp$ to AO', a circumference described from O as center with radius OA will have AO' as a tangent. Hence with O as center and radius equal to the difference of the radii of the given circles, describe a circumference, to which draw a tangent from O'. The radii drawn perpendicular to this tangent will give the points of contact X, Y. As a tangent from O' could be drawn to the other side of the circle to which the other was drawn, a second exterior tangent can be drawn.



429. Let XY be an interior tangent common to Ⓢ O, O'. Join OX, $O'Y$, and draw $O'A \perp$ to OX produced, meeting it in A. AY is obviously a rectangle, and $AX = O'Y$; $\therefore$ $OA = OX + O'Y$, and being $\perp$ to $O'A$, a circumference described from O as center with radius OA will have AO' as tangent. Hence from O as center, with radius equal to the sum of the radii of the given circles; etc.



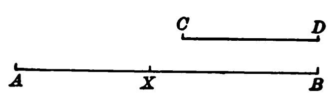
430. Let X be the required point such that $\angle AXB$, formed by tangents XA, XB, to Ⓢ O and O' tangent at P, is equal to a given angle. Join XO, XO', XP. By Ex. 200, $\angle OXO' = \frac{1}{2} \angle AXB$. What is required then is to construct on OO' an isos. triangle having a vertical angle equal to one half the given angle. Hence on OO' describe a segment (311), containing an inscribed angle equal to one half the given angle. The vertex will be the point X.



431. Let ABC be the given triangle, and BC the side to be divided. Produce BA to D, so that $AD = AC$; join DC, and draw $AX \parallel$ to DC. Then $BX : XC = AB : AD = AB : AC$.

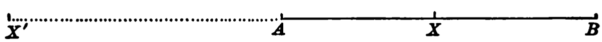
432. Let ABC be the given triangle, and BC the side to be divided. Produce BA indefinitely, and lay off on the part produced $AD = AC$ and $DE = BC$. Join EC, and through A and D draw AX, DY, each $\parallel$ to EC. Then $BX : XY : YC = BA : AD : DE = AB : AC : BC$.

433. Let AB, CD be the given lines, and suppose AB divided in



X so that $\overline{AX}^2 \Bumpeq CD \cdot BX$. Then $\overline{AX}^2 \Bumpeq CD(AB - AX) \Bumpeq CD \cdot AB - CD \cdot AX \Bumpeq \overline{EF}^2 - CD \cdot AX$, EF being a mean proportional between AB and CD. Hence $\overline{AX}^2 + CD \cdot AX \Bumpeq \overline{EF}^2$; $\therefore \overline{AX}^2 + CD \cdot AX + (\frac{1}{2}\,CD)^2 \Bumpeq \overline{EF}^2 + (\frac{1}{2}\,CD)^2 = M^2$, say; hence $AX + \frac{1}{2}\,CD = M$; $\therefore AX = M - \frac{1}{2}\,CD$. Geometrically interpreted this means, find EF, a mean proportional between AB and CD; then find M, the hypotenuse of a rt. $\triangle$ having EF and $\frac{1}{2}\,CD$ as arms; then $AX = M - \frac{1}{2}\,CD$.

Remark.—Note the close correspondence between this geometrical solution of a problem and the algebraic solution of a quadratic equation by completing the square. This method may be applied with facility to the solution of a great variety of problems concerning the sections of a line, a selection of which is appended below for exercise. This method has moreover the advantage of presenting simultaneously the solutions for both the internal and external section. Thus the problem just solved is one case of the more general problem : *Divide a given line in internal or external section so that the square of one segment*; etc. We found above that $(AX + \frac{1}{2}\,CD)^2 \Bumpeq M^2$, M being a line of determined length, but of direction either positive or negative; that is, it may be drawn in either of two opposite directions from a given point A. Hence we may take $AX + \frac{1}{2}\,CD = \pm\, M$, whence $AX = M - \frac{1}{2}\,CD$, $AX' = -M - \frac{1}{2}\,CD$. If then AX,



laid off to the right from A, gives the internal point of section, AX', laid off in the opposite direction, gives the external point of section, and $\overline{AX'}^2 \Bumpeq CD \cdot BX'$.

In the following extra examples, *a* to *g*, it is to be noted that *divide* is to be taken in the general sense of *divide internally or externally*.

a. Divide a given line into segments such that the square of one segment shall be equivalent to the rectangle of the given line and the other segment.

This important problem, the *medial* section, already solved synthetically in Art. 308, is given here as an example of the application of this method, which, it will be perceived, includes both the analysis and the synthesis of the problem.

Let AX be one segment of the given line AB; then $AB - AX$ is the other. By the given conditions, $\overline{AX}^2 \Bumpeq AB(AB - AX) \Bumpeq \overline{AB}^2 - AB \cdot AX$;

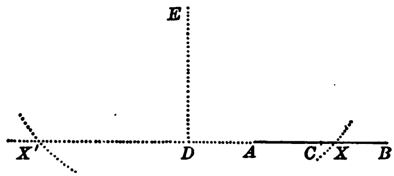
$$\therefore \overline{AX}^2 + AB \cdot AX \Bumpeq \overline{AB}^2;\ \therefore \overline{AX}^2 + AB \cdot AX + (\tfrac{1}{2}AB)^2 \Bumpeq \tfrac{5}{4}\overline{AB}^2;$$

$$\therefore AX + \tfrac{1}{2}AB = \pm\sqrt{5}\,\frac{AB}{2};\ \therefore AX = (\sqrt{5} - 1)\,\frac{AB}{2},\ AX' = -(\sqrt{5} + 1)\,\frac{AB}{2}.$$

The construction given in Art. 308, and that found in Euclid (II. 11) are merely different ways of interpreting the result $AX = (\sqrt{5} - 1)\,\frac{AB}{2}$. To

these we add a third construction, based upon Art. 363, and giving both AX and AX'.

Bisect the given line AB in C, and produce CA to D, so that $AD = AC$. At D draw $DE \perp$ and $=$ to DC or AB. From E as center, with radius



equal to DB, describe an arc cutting AB in X and AD produced in X'. Then

$$\overline{AX}^2 \simeq AB \cdot BX, \text{ and } \overline{AX'}^2 \simeq AB \cdot BX'.$$

$$\text{For } \overline{DX}^2 \simeq \overline{EX}^2 - \overline{DE}^2 \simeq (3\,AC)^2 - (2\,AC)^2 = 5\,\overline{AC}^2;$$

$$\therefore\ DX = \sqrt{5}\frac{AB}{2},\ \therefore\ AX = (\sqrt{5}-1)\frac{AB}{2};$$

$$\text{and } AX' = -(\sqrt{5}+1)\frac{AB}{2}.$$

_b_. Divide a given line into segments such that the sum of their squares shall be equivalent to the rectangle of the given line and one of the segments.

_c_. Divide a given line into segments such that the sum of their squares shall be equivalent to a given square.

N.B. In this and similar examples rectilinear figure may be substituted for square, since a square can be found equivalent to any given rectilinear figure (360).

_d_. Divide a given line into segments such that the difference of their squares shall be equivalent to the rectangle of the given line and a segment.

N.B. The result obtained, $AX = \frac{1}{3}AB$, shows that there is but one, the internal, section.

_e_. Divide a given line into segments such that the difference of their squares shall be equivalent to a given square.

N.B. If AX be a segment of the given line, and $\overline{CD}^2$ the given square, we obtain the equality $2\,AB \cdot AX \simeq \overline{AB}^2 + \overline{CD}^2 \simeq M^2$, and AX is a third proportional to $2\,AB$ and M.

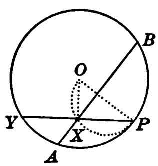
_f_. Divide a given line into segments such that the ratio of their squares is as m to n.

_g_. Divide a given line into segments such that the rectangle of the segments shall be equivalent to a given square.

N.B. On completing the square we obtain $(AX - \frac{1}{2}AB)^2 \simeq \frac{1}{4}\overline{AB}^2 - \overline{CD}^2 \simeq M^2$, say; whence $AX = \frac{1}{2}AB \pm M$, in which result M represents a possible result only when CD is not $> \frac{1}{2}AB$; *i.e.*, the given square must not be greater than the square of one half the given line.

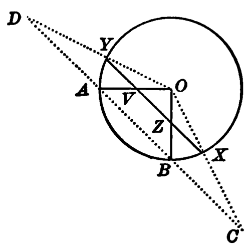
434. Let P be the point through which passes the given chord AB. Draw the radius OPD, at P make $\angle CPD = \angle BPD$, and complete the chord CPE. Then $CP : PE = PB : PA$ (Ex. 198, Ex. 197).

435. Let PXY be the required secant drawn from the given point P and meeting the circumference of $\odot\, O$ in X and Y, so that $PX : XY = m : n$. Draw a tangent PT. Since $XY = \frac{n}{m} PX$, and $PX \cdot PY \simeq \overline{PT}^2$ (303), $PX\left(PX + \frac{n}{m} PX\right) \simeq \frac{m+n}{m} \overline{PX}^2 \simeq \overline{PT}^2$; $\therefore \sqrt{\frac{m+n}{m}} PX = PT$; or $PX : PT = \sqrt{m} : \sqrt{m+n}$, and PX can be found by Art. 363.



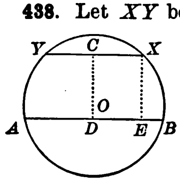
436. Let P be the point in the arc whose chord is AB, and PXY the required chord bisected by AB in X. Join OX, OP; OX is $\perp$ to PY (177), and OXP is a right angle; $\therefore$ a semicircumference described on OP as diameter will pass through X. Hence join OP, on OP as diameter describe a semicircumference; etc.

437. Let OA, OB be the radii perpendicular to each other. Join AB, and produce each way to C and D, so that BC, AD are each equal to AB. Draw OC, OD, cutting the circumference in X and Y

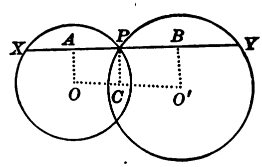


respectively, and join XY. Since $\angle OAB = \angle OBA$ (68), $\angle OAD = \angle OBC$; also OA, $AD = OB$, BC respectively; $\therefore \triangle OAD = \triangle OBC$; $\therefore OD = OC$. But $OY = OX$; $\therefore XY$ is parallel to CD (276); $\therefore XZ = ZV = VY$ (282), since $CB = BA = AD$ (Const.).

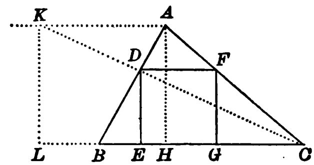
438. Let XY be the required chord $\parallel$ to AB in $\odot\, O$. Draw OC $\perp$ to XY, meeting AB, produced if necessary, in D; draw $XE \perp$ to DB. Since $DE = CX$ (136), $DE : DB = CX : DB = 2\, CX : 2\, DB = XY : AB$. Hence divide DB in E so that $DE : DB$ is the given ratio. Draw $EX \perp$ to DB; etc.



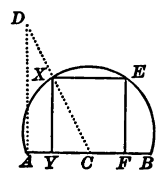
439. Let XY be the secant divided at P in



the given ratio. Join OO'; draw OA, $O'B$, $\perp$ to XY, and draw $PC \parallel$ to OA, meeting OO' in C. CP is also $\perp$ to XY. Since OA, CP, $O'B$, are parallels, we have $CO : CO' = PA : PB = 2\ PA$ or $PX : 2\ PB$ or PY. Hence draw OO' and divide OO' in C in the given ratio; join CP, and through P draw $XPY \perp$ to CP; etc.

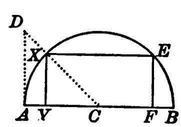


440. Let DG be the required square. Draw the altitude AH to BC, and AK parallel to BC. Join CD and produce to meet AK in K; draw KL perpendicular to CB produced. By similar triangles, $AK : DF = KC : DC$, and $KL : DE = KC : DC$; $\therefore AK : KL = DF : DE$. But $DF = DE$; $\therefore AK = KL = AH$. Hence draw AH perpendicular to BC, and AK parallel to BC and equal to AH; draw KC cutting AB in D; DE is a side of the required square.



441. Let XF be the required square in the segment whose chord is AB. Bisect AB in C, and at A draw AD perpendicular to AB; join CX and produce to meet AD in D. By similar triangles, $DA : AC = XY : CY$. But $XY = YF = 2\ CY$; $\therefore DA = 2\ AC = AB$. Hence at A draw AD perpendicular to AB and equal to AB; bisect AB in C, and draw CD cutting the circumference in X; draw XY perpendicular to AB; XY is a side of the required square.

442. Let XF be the required rectangle inscribed in the semicircle whose diameter is AB. At A draw $AD \perp$ to AB, bisect AB in C, join CX, and produce to meet AD in D. By similar triangles, $AD : AC = XY : YC$; $\therefore AD : 2\ AC$ or $AB = XY : 2\ YC$ or YF. Hence at A draw a perpendicular to AB, and lay off AD so that $AD : AB$ is the ratio of the sides of the given rectangle. Bisect AB in C, and join CD, cutting the circumference in X; draw $XY \perp$ to AB, and $XE \parallel$ to AB; etc.

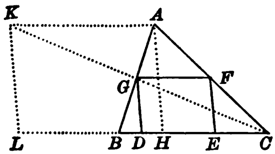


443. Let ABC be the given triangle and DEF the given circle. At any point D of the circumference draw a tangent GDH, at D make $\angle GDF = \angle B$, and $\angle HDE = \angle C$. Join EF; then as $\angle E = \angle GDF = \angle B$, and $\angle F = \angle HDE = \angle C$, $\triangle DEF$ is similar to $\triangle ABC$.

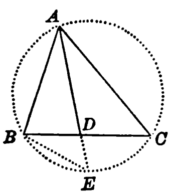
444. Let ABC be the given triangle, and DEF the given circle. Draw radii OD, OE, making an $\angle DOE$ equal to the supplement of $\angle A$, and draw a third radius OF making $\angle FOD$ equal to the supplement of

$\angle B$. Through D, E, F, draw tangents intersecting in H, K, L. The $\triangle DOE$, DOF being respectively the supplements of $\triangle H$ and K, as well as of $\triangle A$ and B, we have $\angle H = \angle A$, $\angle K = \angle B$; etc.

445. Let $DEFG$ be the required par'm inscribed in $\triangle ABC$. From A draw $AH \parallel$ to GD, and $AK \parallel$ to BC; draw CG, and produce to meet AK in K. By similar $\triangle$, $AK : GF = KC : GC$, and $KL : GD = KC : GC$; $\therefore$ $AK : KL = GF : GD$ (249). Hence draw AH making an angle with BC equal to an angle of the given par'm, draw $AK \parallel$ to BC so that $AK : KL$ is the ratio of two adjacent sides of the par'm; draw CK meeting AB in G, draw $GF \parallel$ to BC, and GD, FE, $\parallel$ to AH; etc.



446. Let ABC be the required triangle, having its base BC, and AD the bisector of its vertical angle, given, as also its vertical angle. Describe about ABC a circumference $ABED$, produce AD to E, and join EB. Since $\angle BAE = \angle CAE$, E is the mid point of arc BEC; and since $\angle BAE = \angle CAE = \angle DBE$ (266), and $\angle BED$ is common, $\triangle ABE$ is similar to $\triangle BDE$; $\therefore$ $AE : EB = EB : DE$; $\therefore$ $AE \cdot DE \approx \overline{EB}^2$; $\therefore$ $DE(DE + AD) \approx \overline{EB}^2$; *i.e.*, $\overline{DE}^2 + AD \cdot DE \approx \overline{EB}^2$; $\therefore$ $\overline{DE}^2 + AD \cdot DE + (\frac{1}{2} AD)^2 \approx \overline{EB}^2 + (\frac{1}{2} AD)^2 \approx M^2$, say; $\therefore$ $DE + \frac{1}{2} AD = M$; $\therefore$ $DE + AD = M + \frac{1}{2} AD$. Hence upon BC as base, describe a segment BAC containing an angle equal to the given angle (311), and complete the circumference by arc BEC. Bisect BEC in E, and join EB. Find M, the hypotenuse of a right triangle, whose arms are equal to AD and EB respectively; then with E as center and radius $= M + \frac{1}{2} AD$, describe an arc cutting the circumference in A; join AB, AC, AD; etc.



REMARK.—Two exercises connected with this one, and leading up to it, were accidentally omitted from among the theorems. They are: (a) *The bisectors of all angles inscribed in a given segment meet in a point.* This point, it will be seen, is the mid point of the intercepted arc. (b) *If from the mid point of a given arc any chord be drawn cutting the chord of that arc, the rectangle of the chord thus drawn and its segment intercepted between the arc and its chord, is constant.* Thus we proved above that $AE \cdot DE \approx \overline{EB}^2$, a constant.

LOCI, p. 164.

447. The perpendicular at the mid point of the line joining the given points being the locus of all points equidistant from those points (213),

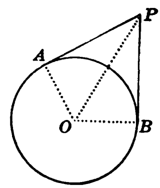
that perpendicular is the locus of the centers of all circumferences passing through those points.

448. The line that passes through the center of the given circle and through the given point, will pass through the center of every circle that is tangent to the given circle at that point (199); hence that line is the required locus.

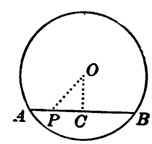
449. The central distance of tangent circles equals the sum or difference of their radii; hence the locus is two circumferences concentric with the given circle, with radii equal to the sum and difference of the radius of the given circle and the given radius.

450. The perpendicular to the given line at the given point passes through the center of every circle that is tangent to the line at that point (192); hence that perpendicular is the required locus.

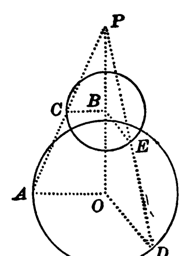
451. The bisectors of the vertical angles formed by intersecting lines constitute the locus of points equidistant from those lines (216); hence they constitute the required locus.



452. Let P be a point in the locus, from which tangents PA, PB, of a given length, are drawn to $\odot$ O. Join OA, OB, OP. Since OA, PA are constants, OP is also constant. Hence the circumference concentric with the given circle and having a radius equal to the hypotenuse of the right triangle having OA, PA as arms, is the required locus.



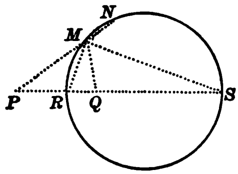
453. Let AB be a chord passing through the given point P in $\odot$ O. Join OP, and draw $OC \perp$ to AB. OCP is a right triangle, and C is a point in the locus. Hence the circumference described on OP as diameter is the required locus.



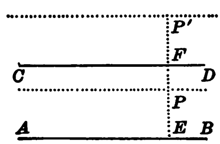
454. Let P be the given point without $\odot$ O. Join PO, and draw radius $OA \perp$ to PO; join PA. Through B, the mid point of PO, draw $BC \parallel$ to OA to meet PA in C. The circumference described from B as center with radius BC, is the required locus. Draw any secant PD; join OD, and through B draw $BE \parallel$ to OD. Then (147) $ED = EP$ and $BE = \frac{1}{2}\ OD = \frac{1}{2}\ OA = BC$ (148); $\therefore$ E, the mid point of PD, is a point in the circumference of $\odot$ B. It may easily be shown that that circumference bisects also the external segments of the secants.

455. On the given base describe a segment containing an angle equal to the given angle; the arc of this segment is the required locus.

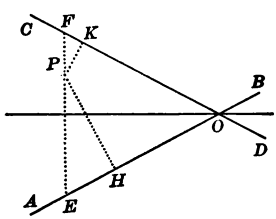
456. Let P, Q be the given points, and M a point in the required locus. Draw PMN, PQ, and MQ, and bisect $\angle$s QMP, QMN by MR, MS respectively, meeting PQ and PQ produced in R and S. Then R is a second point in the locus, since $RP : RQ = MP : MQ$ (278); and S is a third point (280), since $SP : SQ = MP : MQ = RP : RQ$; so that PQ is divided harmonically in R and S in the given ratio. Also RMS is a right angle (Ex. 113); $\therefore$ a circumference described upon RS as diameter is the required locus. Hence divide PQ harmonically in R and S, in the given ratio (309), so that $SP : SQ = RP : RQ$; etc.



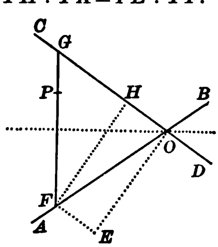
457. (1) If the given lines AB, CD are parallel, draw an intercept EF perpendicular to them, and divide it internally in P, and externally in P', in the given ratio, so that $PE : PF = P'E : P'F$. The parallels drawn through P and P' to the given lines will be the required locus, which, with the given lines, divides the intercept harmonically.

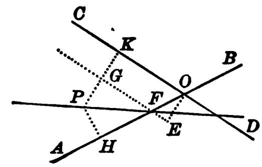


(2) If the given lines intersect in O, draw the intercept EF perpendicular to the bisector of AOC, one of the angles formed, and divide EF at P in the given ratio. A line that passes through P and O is part of the required locus. Draw PH, PK perpendicular to AB, CD respectively; then, by similar triangles, $PH : PK = PE : PF$. By proceeding similarly with the bisector of $\angle AOD$, we can obtain the other part of the locus.

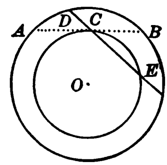


458. Let AB, CD intersect in O. At O draw $OE \perp$ to OC, a side of AOC, one of the angles formed, and make OE equal to the given sum. Through E draw $EF \parallel$ to OG to meet OA in F, and draw $FG \perp$ to the bisector of $\angle AOC$. FG is part of the required locus. From F draw the altitude FH to OG, an arm of $\triangle OFG$ which is isosceles (Ex. 118). FH is equal to OE (136), and the sum of the altitudes from any point P to the arms OA, OC, is equal to FH (Ex. 149), is equal to OE, is equal to the given line. The rest of the locus is found in a similar way.





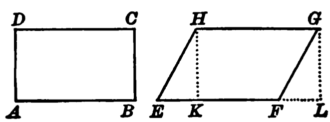
459. Let AB, CD intersect in O. At O draw OE perpendicular to OC, a side of AOC, one of the angles formed, and make OE equal to the given difference. Through E draw EF parallel to OC to meet OA in F, and draw FP the bisector of $\angle AFG$. FP is part of the required locus. From P, any point in FP, draw PH, PK perpendicular to OA, OC respectively. Then $PH = PG$ (101), and $PK = PG + GK = PH + OE$. The rest of the locus may be found in a similar way.



460. In the given $\odot$ O place a chord AB equal to twice a side of the given square, and find its mid point C. The circumference described from O as center, with radius OC, is the required locus. For this circumference is the locus of the mid points of all chords in $\odot$ O equal to AB (218). Now if any secant DE be drawn through C, then rect. $DC \cdot CE \approx$ rect. $AC \cdot CB \approx \overline{AC}^2 \approx$ a square equal to the given square.

EXERCISES, pp. 171–185.

461. Since $AD = BC$, and $EF = BC$, $EF = AD$; $\therefore$ $EA = FD$ (Ax. 2); also $FC = BE$; $\therefore$ rt. $\triangle AEB =$ rt. $\triangle DFC$; etc. (327).



462. Let AC be the rectangle and EG the par'm. Draw HK, $GL \perp$ to EF; then rect. $KG =$ rect. AC (317), and $\approx$ par'm EG (327); $\therefore$ par'm $EG \approx$ rect. AC.

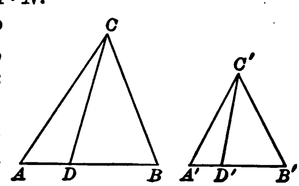
463. Since $AB = AE + EB$, rect. $AB \cdot (AE - EB) \approx$ rect. $(AE + EB)(AE - EB) \approx \overline{AE}^2 - \overline{EB}^2$ (337).

464. Since $AE = AB - EB$, rect. $AE\ (AB + EB) \approx$ rect. $(AB - EB)(AB + EB) \approx \overline{AB}^2 - \overline{EB}^2$.

465. $\angle D$ must be equal to $180° - 62°\ 54'\ 23'' = 117°\ 5'\ 37''$.

466. Let M and N denote the two lines; then

$$(M + N)^2 - (M - N)^2 \approx M^2 + N^2 + 2\,M \cdot N - (M^2 + N^2 - 2\,M \cdot N) \approx 4 \text{ rect. } M \cdot N.$$



467. Since $\angle$s A and ADC are equal to $\angle$s A' and $A'D'C'$ respectively (Const.), $\triangle$s ACD, $A'C'D'$ are similar; $\therefore$ $CD : C'D' = AC : A'C'$; etc.

468. If $\triangle ABC : \triangle A'B'C' = \overline{AB}^2 : \overline{A'B'}^2 = 2:1$, then $AB : A'B' = \sqrt{2} : 1$ (346).

469. Since $AD : A'D' = AB : A'B' = m : n$, $\overline{AD}^2 : \overline{A'D'}^2 = m^2 : n^2$.

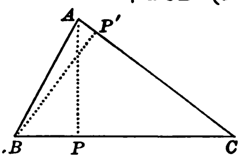
470. If $P : P' = m : n$, then $AC : A'C' = \sqrt{m} : \sqrt{n}$ (346).

471. In the diagram for Art. 349, left-hand figure, suppose BC drawn $\perp$ to AP; then $BC = PQ$, and $CP = BQ$; $\therefore AC = AP - BQ$; $\therefore \overline{AB}^2 - \overline{PQ}^2 \simeq \overline{AB}^2 - \overline{BC}^2 \simeq \overline{AC}^2 \simeq (AP - BQ)^2$.

472.
$$\overline{AB}^2 + \overline{BC}^2 \simeq \overline{AE}^2 + \overline{EB}^2 + 2\,AE \cdot EB + \overline{CE}^2 + \overline{EB}^2$$
$$\simeq \overline{AE}^2 + \overline{CE}^2 + 2\,\overline{EB}^2 + 2\,AE \cdot EB$$
$$\simeq \overline{AC}^2 + 2\,AB \cdot EB.$$

Draw $DF \perp$ to AB; then EF or $CD = AE - AF$, and $DF = CE$;
$$\therefore \overline{CD}^2 + \overline{AD}^2 \simeq \overline{AE}^2 + \overline{AF}^2 - 2\,AE \cdot AF + \overline{CE}^2 + \overline{AF}^2$$
$$\simeq \overline{AC}^2 - 2\,AF \cdot (AE - AF) \simeq \overline{AC}^2 - 2\,CD(AE - CD).$$

473.
$$\overline{AB}^2 + \overline{BC}^2 - (\overline{CD}^2 + \overline{AD}^2) \simeq \overline{AC}^2 + 2\,AB \cdot EB - \overline{AC}^2$$
$$+ 2\,CD \cdot (AE - CD) \simeq 2\,AB(AB - AE) + 2\,CD(AE - CD).$$



474. Draw $BP' \perp$ to AC; then (350) $\overline{AB}^2 \simeq \overline{AC}^2 + \overline{BC}^2 - 2\,AC \cdot CP' \simeq \overline{BC}^2 + \overline{AC}^2 - 2\,BC \cdot CP$; $\therefore AC \cdot CP' \simeq BC \cdot CP$.
(Ax. 1, Ax. 3).

475. If $\overline{AB}^2 \simeq \overline{AC}^2 + 3\,\overline{PC}^2$, it is equivalent to $\overline{AP}^2 + 4\,\overline{PC}^2$, since $\overline{AC}^2 = \overline{AP}^2 + \overline{PC}^2$, while $\overline{AB}^2 \simeq \overline{AP}^2 + \overline{BP}^2$; then $\overline{BP}^2 = 4\,\overline{PC}^2$; $\therefore BP = 2\,PC$; $\therefore P$ trisects BC.

476. In the diagram for Ex. 474, since $\triangle$s APC, $BP'C$ are similar, $AP : BP' = CP : CP' = AC : BC$.

477. Draw $CF \perp$ to AB produced; then $AF = AB + BF$, and $BF = \frac{1}{2}BC$, since, $\angle CBF$ being 60° and $\angle F$ a right angle, CBF is half of an equilateral triangle. $\overline{AC}^2 \simeq \overline{AF}^2 + \overline{CF}^2 \simeq \overline{AB}^2 + \overline{BF}^2 + 2\,AB \cdot BF + \overline{CF}^2$. But $\overline{BF}^2 + \overline{CF}^2 \simeq \overline{BC}^2$, and $2\,AB \cdot BF \simeq AB \cdot BC$; $\therefore \overline{AC}^2 \simeq \overline{AB}^2 + \overline{BC}^2 + AB \cdot BC$.

478. Draw $CF \perp$ to AB; then $AF = AB - BF = \frac{1}{2}BC$. $\overline{AC}^2 \simeq \overline{AF}^2 + \overline{CF}^2 \simeq \overline{AB}^2 + \overline{BF}^2 - 2\,AB \cdot BF + \overline{CF}^2 \simeq \overline{AB}^2 + \overline{BC}^2 - AB \cdot BC$, as above.

479. Let BC be the base of an isosceles $\triangle ABC$; draw $CD \perp$ to AB; then BD is the projection of BC on AB, and $\overline{BC}^2 \simeq \overline{AB}^2 + \overline{AC}^2 - 2\,AB \cdot AD \simeq 2(\overline{AB}^2 - AB \cdot AD) \simeq 2\,AB \cdot BD$.

480. Let AC, BD, the diagonals of parallelogram $ABCD$, intersect in O. Since $AO = CO$, $\overline{AB}^2 + \overline{BC}^2 \simeq 2\,\overline{AO}^2 + 2\,\overline{BO}^2$. Similarly $\overline{AD}^2 + \overline{CD}^2 \simeq 2\,\overline{AO}^2 + 2\,\overline{DO}^2$ or $2\,\overline{BO}^2$; $\therefore \overline{AB}^2 + \overline{BC}^2 + \overline{AD}^2 + \overline{CD}^2 \simeq 4\,\overline{AO}^2 + 4\,\overline{BO}^2 \simeq \overline{AC}^2 + \overline{BD}^2$.

481. In $\triangle ABC$ let AD be the median drawn to BC. $\triangle ADB \simeq \triangle ADC$, since they have equal bases, DB, DC (Hyp.), and the same altitude (316).

482. In $\triangle ABC$, let AD, BE, CF, be the medians drawn to BC, AC, AB, respectively. Now $\overline{AB}^2 + \overline{AC}^2 \simeq 2\,(\overline{AD}^2 + \overline{BD}^2)$, $\overline{AB}^2 + \overline{BC}^2 \simeq$

$2(\overline{AE}^2+\overline{BE}^2)$, and $\overline{AC}^2+\overline{BC}^2 \bumpeq 2(\overline{AF}^2+\overline{CF}^2)$; $\therefore$ $2(\overline{AB}^2+\overline{AC}^2+\overline{BC}^2) \bumpeq 2(\overline{AD}^2+\overline{BE}^2+\overline{CF}^2)+\frac{1}{2}(\overline{AB}^2+\overline{AC}^2+\overline{BC}^2)$; $\therefore$ $3(\overline{AB}^2+\overline{AC}^2+\overline{BC}^2) \bumpeq 4(\overline{AD}^2+\overline{BE}^2+\overline{CF}^2)$.

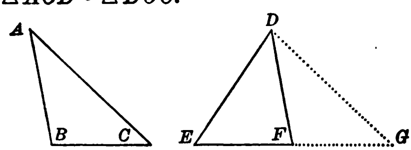
483. Since $2\,\overline{BC}^2 \bumpeq 2(\overline{AB}^2+\overline{AC}^2-AB\cdot AE-AC\cdot AD)$, $\overline{BC}^2 \bumpeq AB(AB-AE)+AC(AC-AD) \bumpeq AB\cdot BE+AC\cdot CD$.

484. If $AB>AC>BC$, then $(AB-AC)^2+(AB-BC)^2+(AC-BC)^2>0$; $\therefore$ $\overline{AB}^2+\overline{AC}^2-2\,AB\cdot AC+\overline{AB}^2+\overline{BC}^2-2\,AB\cdot BC+\overline{AC}^2+\overline{BC}^2-2\,AC\cdot BC>0$; $\therefore$ $2(\overline{AB}^2+\overline{AC}^2+\overline{BC}^2)>2(AB\cdot AC+AB\cdot BC+AC\cdot BC)$; $\therefore$ $\overline{AB}^2$ + etc. $> AB\cdot AC$ + etc.

485. Let AD be the median to hypotenuse BC of rt. $\triangle ABC$. Since $AD=\frac{1}{2}BC$ (Ex. 144), $\overline{AD}^2=\frac{1}{4}\overline{BC}^2$.

THEOREMS, p. 186.

486. Let the diagonals AC, BD of parallelogram $ABCD$ intersect in O. Since $OB=OD$, $\triangle AOB \bumpeq \triangle AOD$ (332). Similarly $\triangle AOB \bumpeq \triangle BOC$, and $\triangle AOD \bumpeq \triangle DOC$.



487. In $\triangle$s ABC, DEF let AB, $BC = DF$, FE respectively, and $\angle B$ be supplement of $\angle F$. Produce EF to G, so that $FG=BC$, and join DG. Since $\angle DFG$ is supplement of $\angle F$, $\angle DFG=\angle B$; $\therefore$ $\triangle DFG=\triangle ABC$ (66). But since $FG=BC=EF$, $\triangle DFG \bumpeq \triangle DEF$ (332); $\therefore$ $\triangle ABC \bumpeq \triangle DEF$.

488. Let P be the given point in par'm $ABCD$. Through P draw $QPR \parallel$ to AB. $\triangle PAB \bumpeq \frac{1}{2}$ par'm AR (331), and $\triangle PCD \bumpeq \frac{1}{2}$ par'm DR; $\therefore$ $\triangle PAB+\triangle PCD \bumpeq \frac{1}{2}(AR+DR) \bumpeq \frac{1}{2}$ par'm $ABCD$. Similarly $\triangle PAD+\triangle PBC \bumpeq \frac{1}{2}$ par'm $ABCD$; etc.

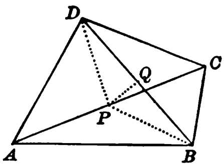
489. Let P be the point in diagonal AC of par'm $ABCD$, through which EPF, GPH are drawn $\parallel$ to AB, BC respectively. Since (140) $\triangle ABC \bumpeq \triangle ACD$, $\triangle PAG \bumpeq \triangle PAE$, and $\triangle PCF \bumpeq \triangle PCH$, $\triangle ABC-(\triangle PAG+\triangle PCF) \bumpeq \triangle ACD-(\triangle PAE+\triangle PCH)$; $\therefore$ par'm $PB \bumpeq$ par'm PD.

490. Let EF join the mid points E, F of the base or parallel sides of trapezoid $ABCD$. Join FA, FB. Since (332) $\triangle FEA \bumpeq \triangle FEB$, and $\triangle AFD \bumpeq \triangle BFC$, $\triangle FEA+\triangle AFD \bumpeq \triangle FEB+\triangle BFC$; *i.e.*, $AEFD \bumpeq BEFC$.

491. Let EF, FG, GH, HE, join the mid points of the sides AB, BC, CD, DA, of the quadrilateral $ABCD$. Draw the diagonals AC,

BD. Then (148) EF and GH are each parallel to AC and equal to $\frac{1}{2}$ AC; $\therefore$ $EFGH$ is a parallelogram. Also $\triangle CFG \simeq \frac{1}{4} \triangle CBD$; etc. Hence $\triangle CFG + \triangle BEF + \triangle AEH + \triangle DGH \simeq \frac{1}{2} ABCD$.

492. Let P be the mid point of AC, a diagonal of trapezium $ABCD$. Join PB, PD. Since $\triangle BPA \simeq \triangle BPC$, and $\triangle DPA \simeq \triangle DPC$, $\triangle BPA + \triangle DPA \simeq \triangle BPC + \triangle DPC$.



493. Join P, Q, the mid points of the diagonals AC, BD, and draw PB, PD. Since $\overline{AB}^2 + \overline{BC}^2 \simeq 2(\overline{PA}^2 + \overline{PB}^2)$, and $\overline{CD}^2 + \overline{DA}^2 \simeq 2(\overline{PA}^2 + \overline{PD}^2)$, $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 \simeq 4\,\overline{AP}^2 + 2\,\overline{PB}^2 + 2\,\overline{PD}^2$. Now $4\,\overline{AP}^2 \simeq \overline{AC}^2$, and $2(\overline{PB}^2 + \overline{PD}^2) \simeq 4(\overline{BQ}^2 + \overline{PQ}^2) \simeq \overline{BD}^2 + 4\,\overline{PQ}^2$; $\therefore$ $\overline{AB}^2 +$ etc. $\simeq \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{PQ}^2$.

494. Draw the diagonal BD intersecting AC in O. Then $\triangle POD \simeq \triangle POB$, $\triangle COD \simeq \triangle COB$; $\therefore$ $\triangle POD - \triangle COD \simeq \triangle POB - \triangle COB$; $\therefore$ $\triangle PCD \simeq \triangle PCB$.

495. Through P draw $PQR \parallel$ to AB, to meet AD, BC produced in Q and R respectively. Then $\triangle PAB \simeq \frac{1}{2}$ par'm $AR \simeq \frac{1}{2}$ (par'm $AC +$ par'm DR) ; also $\triangle PDC \simeq \frac{1}{2}$ par'm DR; $\therefore$ $\triangle PAB - \triangle PCD \simeq \frac{1}{2}$ par'm AC.

Remark. — Compare Ex. 488, which might be combined with this one by changing "without," in this, to "within or without," and $-$ to $\pm$.

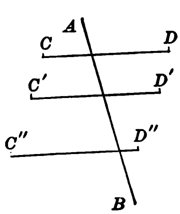
496. Let the diagonals AC, BD of trapezoid $ABCD$ intersect in O. Then

(1) $\triangle DAB \simeq \triangle CBA$ (332) ; $\therefore$ $\triangle DAB - \triangle OAB \simeq \triangle CBA - \triangle OAB$; $\therefore$ $\triangle OAD \simeq \triangle OBC$.

(2) Since $\angle ACD =$ alternate $\angle CAB$, and $\angle CDB =$ alternate $\angle DBA$, $\triangle OAB$ is similar to $\triangle OCD$; $\therefore$ $\triangle OAB : \triangle OCD = \overline{AB}^2 : \overline{CD}^2$.

497. Let ABC, DEF be triangles having $\angle A$ equal to $\angle D$. Then $BA \cdot AC : ED \cdot DF = \triangle ABC : \triangle DEF$ (341). But $\triangle ABC \simeq \triangle DEF$ (Hyp.); $\therefore$ $BA \cdot AC = ED \cdot DF$; $\therefore$ $AB : DE = DF : AC$.

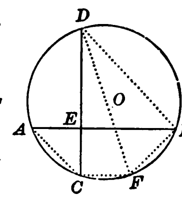
498. Let AC, BD be diagonals of the quadrilateral $ABCD$. Through A and C draw EF, GH, each parallel to BD; and through B and D draw FG, EH, parallel to AC; EG is a parallelogram (Const.), as are also BE and BH. Now $BE \simeq 2 \triangle ABD$, and $BH \simeq 2 \triangle CBD$; $\therefore$ $BE + BH \simeq 2(\triangle ABD + \triangle CBD)$; *i.e.*, parallelogram $EG \simeq 2\,ABCD$.



Remark. — This exercise may be employed to give a class some notion of series of equivalent figures. Thus if a line AB be drawn on the black-

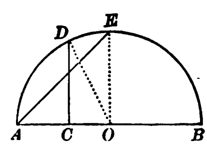
board, and a ruler CD be moved so as always to make the same angle with AB, it will be easily seen that each of the infinite series of quadrilaterals that can be formed by joining AC, AD, BC, BD, is equal to one half of a certain parallelogram. Or draw CD and move AB.

499. In $\odot O$ let C, D be points equally distant from O in diameter AB. Draw any chord ECF, and join D and O each with E and F. Since $\overline{EC}^2+\overline{ED}^2 \bumpeq 2(\overline{OE}^2+\overline{OC}^2)$, and $\overline{FC}^2+\overline{FD}^2 \bumpeq 2(\overline{OF}^2+\overline{OC}^2)$, $\overline{ED}^2+\overline{FD}^2+\overline{EC}^2+\overline{FC}^2 \bumpeq 4(\overline{OE}^2+\overline{OC}^2)$; $\overline{ED}^2+\overline{FD}^2+\overline{EC}^2+\overline{FC}^2+2EC\cdot CF \bumpeq 4\overline{OE}^2+4\overline{OC}^2+2EC\cdot CF$. Now $\overline{EC}^2+\overline{FC}^2+2EC\cdot CF \bumpeq \overline{EF}^2$, $4\overline{OE}^2 \bumpeq \overline{AB}^2$, $4\overline{OC}^2 \bumpeq \overline{CD}^2$, and $2EC\cdot CF \bumpeq 2AC\cdot CB$; $\therefore \overline{ED}^2+\overline{FD}^2+\overline{EF}^2 \bumpeq \overline{AB}^2+\overline{CD}^2+2AC\cdot CB$, a constant.



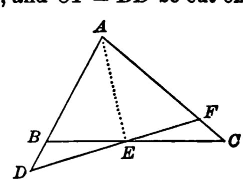
500. In $\odot O$ let AB, CD be chords perpendicular to each other at E. Join AC, BD, and draw CF parallel to AB, meeting the circumference in F. Join FB, FD. Since DCF is a right angle (107), DBF is a semicircumference and DF a diameter. Also $AC = BF$ (195). Since $\overline{AE}^2+\overline{EC}^2 \bumpeq \overline{AC}^2 \bumpeq \overline{BF}^2$, and $\overline{BE}^2+\overline{DE}^2 \bumpeq \overline{BD}^2$, $\overline{AE}^2+\overline{EC}^2+\overline{BE}^2+\overline{DE}^2 \bumpeq \overline{BF}^2+\overline{BD}^2 \bumpeq \overline{DF}^2$.

501. In $\odot O$ let AB, CD be any two chords intersecting in P at a given distance OP from O. Through P draw the diameter EPF. Since EF and OP are constants, PE and PF are also constants. Now $PA\cdot PB \bumpeq PC\cdot PD \bumpeq PE\cdot PF$, a constant.



502. Let CD be a side of the square inscribed in semicircle ADB, and AE a side of the square inscribed in $\odot ADB \cdots$. Join OD, OE. Since $CD=2\,CO$, $\overline{CD}^2+\overline{CO}^2$ or $5\,\overline{CO}^2 \bumpeq \overline{OD}^2 = \overline{OE}^2$. But $\overline{AE}^2 = 2\,\overline{OE}^2$; $\therefore\ \overline{CD}^2 : \overline{AE}^2 = 2:5$.

503. In $\triangle ABC$ let AB be produced to D, and $CF = BD$ be cut off from AC, and let DF joining D and F cut BC in E. Join AE. By Art. 341,



$$\triangle BED : \triangle CEF = BE\cdot ED : CE\cdot EF;$$

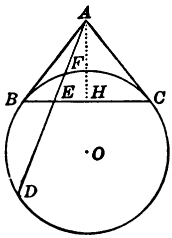
also $\triangle AEB : \triangle AEC = BE : CE$;

$$\therefore\ \frac{\triangle BED}{\triangle AEB} : \frac{\triangle CEF}{\triangle AEC} = ED : EF;$$

also $\dfrac{\triangle BED}{\triangle AEB} = \dfrac{BD}{AB}$, and $\dfrac{\triangle CEF}{\triangle AEC} = \dfrac{CF}{AC}$; $\therefore\ \dfrac{BD}{AB} : \dfrac{CF}{AC} = ED : EF$;

$\therefore\ \dfrac{1}{AB} : \dfrac{1}{AC} = ED : EF$, since $BD = CF$; $\therefore\ ED : EF = AC : AB$.

504. Let tangents AB, AC meet $\odot O$ in B, C respectively, and secant AD meet the circumference in F and D, and cut chord BC in E. Draw $AH \perp$ to BC.



$$\overline{AB}^2 \bumpeq \overline{AE}^2 + \overline{BE}^2 + 2\,BE \cdot EH;$$

$$\therefore \overline{AB}^2 - \overline{AE}^2 \bumpeq \overline{BE}^2 + 2\,BE \cdot EH \bumpeq BE \cdot EC$$
$$\bumpeq EF \cdot DE.$$

Again $\overline{AB}^2 \bumpeq AD \cdot AF \bumpeq (AE + DE)(AE - EF)$
$$\bumpeq \overline{AE}^2 + AE \cdot DE - AD \cdot EF;$$
$$\therefore \overline{AB}^2 - \overline{AE}^2 \bumpeq AE \cdot DE - AD \cdot EF;$$
$$\therefore EF \cdot DE \bumpeq AE \cdot DE - AD \cdot EF; \quad \therefore EF(AD + DE) \bumpeq AE \cdot DE;$$
$$\therefore AD + DE : DE = AE : EF; \quad \therefore AD + DE - DE : DE$$
$$= AE - EF : EF; \text{ i.e., } AD : DE = AF : EF.$$

QUESTIONS, p. 194.

505. An equilateral triangle has one altitude; an isosceles triangle, two; a scalene triangle, three; a square, one; a rectangle (not a square), two; a trapezoid, two or three; a trapezium, three or four.

506. The formula on p. 192 becomes, for the equilateral triangle, $h = \frac{a}{2}\sqrt{3}$, here $= 3\sqrt{3}$.

507. Denoting an arm by b, and the base by a, the formula becomes for the isosceles triangle, $h = \sqrt{b^2 - \left(\frac{a}{2}\right)^2}$, here $= \sqrt{260} = 16.12$ for alt. on a; and $h = \frac{a}{b}\sqrt{b^2 - \left(\frac{a}{2}\right)^2} = \frac{16}{18}\sqrt{260} = 14.33$ for alt. on b.

508. Denoting 30, 60, 40, by a, b, c, respectively, since $h = \frac{2\,S}{a}$, a being the side to which the alt. h is drawn, we have alt. on $a = \frac{180 \times 2}{30} = 12$; alt. on $b = \frac{180 \times 2}{60} = 6$; alt. on $c = \frac{180 \times 2}{40} = 9$.

509. Denoting by a, b, c, respectively, the sides on which the altitudes 8, 12, 14, fall respectively, since $a = \frac{2\,S}{h}$, we have

$$a = \frac{252 \times 2}{8} = 63; \; b = \frac{252 \times 2}{12} = 42; \; c = \frac{252 \times 2}{14} = 36.$$

510. $S = 65 \times 32 = 2080$; perimeter $= 2(65 + 32) = 194$; diagonal $= \sqrt{65^2 + 32^2} = 72.45$.

511. $S = \frac{1}{2}(23 \times 10) = 115$.

512. $S = 221$ sq. ft.; base $= 5\frac{2}{3} \times 3 = 17$ ft.; $\therefore$ alt. $= 221 \times 2 \div 17 = 26$, and 26 ft. $= 312$ in.

513. Let S, S' be the areas; then $S:S' = 15 \times 8 : 16 \times 10 = 3:4$.

514. Let h, h' be the altitudes; then $\frac{1}{2}h \times 26 = \frac{1}{2}h' \times 36$; $\therefore h:h' = 18:13$.

515. $S = \frac{1}{2}(23 + 17) \times 30 = 600$, and 600 sq. in. $=4\frac{1}{6}$ sq. ft.

516. $\triangle ABC : \triangle ADE = 42 \times 34 : 30 \times 15 = 238 : 75$.

517. If l denote the length, $l^2 = 20^2 + 15^2$; $\therefore l = 25$.

518. If x denote the required segment, $x \times 10 = 12 \times 7$; $\therefore x = 8.4$.

519. Let h, h', h'', denote the altitudes upon the sides whose numerical measures are 12, 15, 9, respectively; then $h = \frac{2}{12}\sqrt{18 \times 6 \times 3 \times 9} = 9$; $h' = \frac{2}{15} \times 54 = 7\frac{1}{5}$; $h'' = \frac{2}{9} \times 54 = 12$.

520. Let m, m', m'', denote the medians to sides 12, 15, 9, respectively;

$$m = \tfrac{1}{2}\sqrt{2(15^2 + 9^2) - 12^2} = \tfrac{1}{2} \times 21.63 = 10.82.$$

$$m' = \tfrac{1}{2}\sqrt{2(12^2 + 9^2) - 15^2} = \tfrac{1}{2} \times 15 = 7.5.$$

$$m'' = \tfrac{1}{2}\sqrt{2(12^2 + 15^2) - 9^2} = \tfrac{1}{2} \times 25.63 = 12.82.$$

521. Let d, d', d'', denote the bisectors to sides 12, 15, 9, respectively;

$$d = \frac{2}{15 + 9}\sqrt{15 \times 9 \times 18 \times 6} = \tfrac{54}{12}\sqrt{5} = 10.08.$$

$$d' = \frac{2}{12 + 9}\sqrt{12 \times 9 \times 18 \times 3} = \tfrac{108}{21}\sqrt{2} = 7.27.$$

$$d'' = \frac{2}{12 + 15}\sqrt{12 \times 15 \times 18 \times 9} = \tfrac{108}{24}\sqrt{10} = 14.23.$$

522. $R = \dfrac{12 \times 15 \times 9}{4} \times \dfrac{1}{\sqrt{18 \times 6 \times 3 \times 9}} = \frac{405}{54} = 7.5.$

523. $S = \sqrt{18 \times 6 \times 3 \times 9} = 54$.

524. $S = \sqrt{8 \times 2 \times 3 \times 3} = 12$.

525. $S = \frac{9}{4}\sqrt{3} = 3.897$.

526. If h denote the altitude, and a a side, then $S = a \times \dfrac{h}{2}$, and $a = \dfrac{2h}{\sqrt{3}}$; $\therefore S = \dfrac{h^2}{\sqrt{3}} = \dfrac{121}{\sqrt{3}} = 69.85$.

527. Let m, m', m'', denote the medians upon sides 25, 24, 7, and h the alt.; then $m = \frac{1}{2}\sqrt{2(24^2 + 7^2) - 25^2} = \frac{25}{2} = 12.5$.

$$m' = \tfrac{1}{2}\sqrt{2(25^2 + 7^2) - 24^2} = \tfrac{1}{2} \times 27.78 = 13.89.$$

$$m'' = \tfrac{1}{2}\sqrt{2(25^2 + 24^2) - 7^2} = \tfrac{1}{2} \times 48.51 = 24.26.$$

$$h = \tfrac{2}{25}\sqrt{28 \times 3 \times 4 \times 21} = \frac{2 \times 84}{25} = 6.7.$$

528. Let x denote the numerical measure of the tangent; then $x^2 = 27 \times 3 = 81$; $\therefore x = 9$.

529. Let y denote the numerical measure of the secant; then $8y = 81$; $\therefore y = 10.125$.

530. Let z denote the numerical measure of the second secant; then $3z = 27 \times 5 = 135$; $\therefore$ $z = 45$; $\therefore$ $27 - 5 = 22$, and $45 - 3 = 42$, are the internal segments.

PROBLEMS, p. 196.

531. At the mid point D of the base BC of the given $\triangle ABC$, erect a perpendicular meeting in E the parallel to BC through A. Join EB, EC; etc.

532. Produce AB, a side of the given square $ABCD$, to E, making BE equal to AB. Join CA, CE; etc.

533. At A in the base of the given square $ABCD$ make an $\angle BAE$ equal to the given angle, and let AE meet DC in E. Through B draw BF parallel to AE to meet DC produced in F; etc.

534. The section may be effected in various ways; probably the easiest is to apply Art. 363 so as to obtain a line divided in the given ratio, then to divide the given line in that ratio, as follows: on AM, one of two indefinite lines perpendicular at A, lay off $A - 2 = (7 - 5) = 2$ parts of any convenient length, also $A - 10 = 5 \times 2 = 10$, and $A - 12 = 7 + 5 = 12$, such parts. With 2 as center, and radius equal to $A - 12$, describe an arc cutting AN in B. Then $\overline{AB}^2 \Bumpeq 12^2 - 2^2 \Bumpeq 140$; $\therefore$ $\overline{AB}^2 : \overline{A - 10}^2 = 140 : 100 = 7 : 5$; $\therefore$ $AB : A - 10 = \sqrt{7} : \sqrt{5}$, and we can now divide the given line in the ratio of AB to $A - 10$ by Art. 304, making use of the lines AM, AN already drawn.

535. Let $ABCD$ be the given parallelogram, and O the intersection of its diagonals. (1) Through P and O draw a line meeting two parallel sides in Q and R. (2) Through O draw a perpendicular to two parallel sides meeting them in Q and R. (3) Through O draw a parallel to the given line meeting two parallel sides in Q and R. The intercept QR bisects the parallelogram (Ex. 85).

536. Let ABC be the given triangle; draw the median AM, join PM, and draw AN parallel to PM. $\triangle APM \Bumpeq \triangle NMP$ (332), $\triangle PBM + \triangle NMP \Bumpeq \triangle PBM + \triangle APM$. But $\triangle PBM + \triangle APM = \triangle ABM \Bumpeq \frac{1}{2} \triangle ABC$; $\therefore$ $PBN \Bumpeq \frac{1}{2} ABC$.

537. Proceed as in Ex. 536, but instead of taking BM equal to $\frac{1}{2}$ BC, take BM equal to $\dfrac{1}{n} BC$; etc.

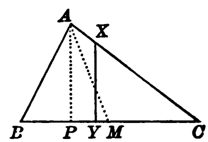
538. Let $ABCD$ be the given quadrilateral; join AC, and through D draw $DE \parallel$ to AC, meeting BA produced in E. $\triangle CEB \Bumpeq ABCD$ (358); hence CF, drawn to the mid point of EB, bisects the trapezium.

N.B. — The vertex can always be chosen so that the mid point of EB will lie in a side.

539. Proceed as in Ex. 538, but instead of taking BF equal to $\frac{1}{2}$ BE, take BF equal to $\frac{1}{n}$ BE; etc.

REMARK. — It is evident that by processes similar to those indicated in Ex. 537 and Ex. 539, triangles, or trapeziums, can be divided into parts having any ratio, $m:n$, $\sqrt{m}:\sqrt{n}$, etc., by lines drawn through a given point P in a side. If P is at the extremity of a side, *i.e.*, at a vertex, the process becomes much simplified.

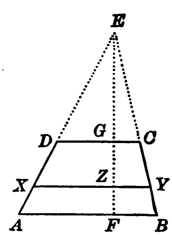
540. Upon AB, a side of the given $\triangle ABC$, describe a semicircumference ANB, and through the mid point of AB draw a $\perp$ to meet the arc in N. Join AN, and lay off on AB, $AD = AN$. Then DE, drawn parallel to BC, bisects $\triangle ABC$. For AN or $AD : AB = 1 : \sqrt{2}$; $\therefore$ $\triangle ADE : \triangle ABC = 1 : 2$ (342).



541. Let XY be the required perpendicular in $\triangle ABC$. Draw the altitude AP and the median AM. Now since $\triangle XYC \approx \frac{1}{2} \triangle ABC \approx \triangle AMC$, $\triangle APC : \triangle XYC = \triangle APC : \triangle AMC = PC : MC$. But also $\triangle APC : \triangle XYC = \overline{PC}^2 : \overline{YC}^2$ (342); $\therefore$ $\overline{PC}^2 : \overline{YC}^2 = PC : MC$, or $PC : \overline{YC}^2 = 1 : MC$; $\therefore$ $\overline{YC}^2 = PC \cdot MC$. Hence we find a mean proportional YC between PC and MC; etc.

REMARK. — It is evident that by processes similar to those given in Ex. 539 and Ex. 540, a triangle can be divided into parts having the ratio of $m : n$, $\sqrt{m} : \sqrt{n}$, etc., by a line $\parallel$ to or $\perp$ to a side. In dividing AB into parts having the ratio $\sqrt{m} : \sqrt{n}$, the process of Art. 363 would be employed instead of the special construction employed when the ratio is $\sqrt{2} : 1$.

542. The required point is the intersection of the three medians. It is easily shown that the three triangles having their vertices at this point, and having as bases the sides of the given triangle, are equal (see diagram for Ex. 154).



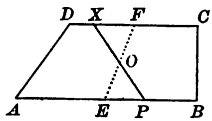
543. Let AB, CD be the bases, and XY the required parallel. Produce AD, BC to meet in E, and draw the altitude EF, cutting CD in G and XY in Z.

$$\triangle EAB : \triangle EDC = \overline{EF}^2 : \overline{EG}^2;$$
$$\therefore\ \triangle EAB : \triangle EAB - \triangle EDC$$
$$\text{or } ABCD = \overline{EF}^2 : \overline{EF}^2 - \overline{EG}^2.$$
$$\triangle EAB : \triangle EXY = \overline{EF}^2 : \overline{EZ}^2;$$
$$\therefore\ \triangle EAB : \triangle EAB - \triangle EXY \text{ or } ABYX = \overline{EF}^2 : \overline{EF}^2 - \overline{EZ}^2;$$
$$\therefore\ ABYX : ABCD \text{ or } 2\,ABYX = \overline{EF}^2 - \overline{EZ}^2 : \overline{EF}^2 - \overline{EG}^2;$$

$\therefore$ $\overline{EF}^2 - \overline{EG}^2 \approx 2\,\overline{EF}^2 - \overline{EZ}^2$; $\therefore$ $2\,\overline{EZ}^2 \approx \overline{EF}^2 + \overline{EG}^2 \approx \overline{BM}^2$, say; $\therefore$ $EZ : BM = 1 : \sqrt{2}$. We accordingly find BM, the side of a square,

equivalent to the sum of $\overline{EF}^2$ and $\overline{EG}^2$; from this we find EZ; XY drawn through Z parallel to AB bisects the trapezoid.

544. Let P be the point in the base AB of trapezoid $ABCD$. Join the mid points E, F of AB and CD. Then EF bisects $ABCD$ (338). Through O, the mid point of EF, draw POX to meet CD in X. $\triangle FOX \approx \triangle POE$ (63); $\therefore POFCB + FOX \approx POFCB + POE \approx BEFC \approx \frac{1}{2} ABCD$.



545. Let AD be the altitude on BC of the given $\triangle ABC$. On BC lay off BE equal to the given line; produce AD to F, so that $DF : DA = BC : BE$. Join FB, FE; $\frac{1}{2} DF \cdot BE \approx \frac{1}{2} DA \cdot BC$.

546. Let AD be the altitude on BC of the given $\triangle ABC$. On DC lay off DE equal to the given line, and produce AD to F, so that $DF : DA = BC : DE$, etc., as in Ex. 545.

547. Let AD be the altitude on BC of the given $\triangle ABC$. Take EF equal to the given hypotenuse, and on it describe a semicircumference EHF. At E draw $EG \perp$ to EF, and take EG so that $EG : AD = BC : EF$. Through G draw $GH \parallel$ to EF to meet the circumf. in H; join HE, HF, and draw $HK \parallel$ to EG. Then $\frac{1}{2} HK \cdot EF \approx \frac{1}{2} AD \cdot BC$.

548. Construct, as in Ex. 547, a right triangle equivalent to half the given triangle, and having the hypotenuse equal to the given length. This right triangle will be half of the required isosceles triangle.

549. Let AD be the altitude on BC of the given $\triangle ABC$. At the mid point G of the given base EF, erect a perpendicular GH so that $GH : AD = BC : EF$; etc.

550. Let AD be the altitude on BC of the given $\triangle ABC$. At A make $\angle DAE = 30°$, and let AE meet DC in E. Then $AE = AD \times \frac{2}{\sqrt{3}}$. Now find a mean proportional XY between AE and BC; the equilateral $\triangle ZXY$ constructed on the base $XY \approx \overline{XY}^2 \times \frac{\sqrt{3}}{4}$ (368, 5°, Scholium) $\approx AE \cdot BC \times \frac{\sqrt{3}}{4} \approx AD \times \frac{2}{\sqrt{3}} \times BC \times \frac{\sqrt{3}}{4} \approx \frac{1}{2} AD \cdot BC$.

551. Let $ABCD$ be the given square. Produce AB to E, making BE equal to BA, and join CA, CE. Then construct, as in Ex. 550, an equilateral triangle equivalent to $\triangle CAE$.

552. Construct a right triangle whose arms are respectively equal to homologous bases of the given triangles. The triangle similar to each of these, constructed on the hypotenuse of the right triangle as homologous base, will be the triangle required (348).

553. Construct a right triangle having the hypotenuse and an arm respectively equal to homologous bases of the given triangles; the simi-

lar triangle constructed on the other arm as homologous base will be the triangle required.

554. Find EF, a mean proportional between AD, the altitude on BC, and BC, the base of the given $\triangle ABC$. $\overline{EF}^2 \approx \triangle ABC$. Then by Art. 363 find a square X^2 such that $X^2 : \overline{EF}^2 = 5 : 3$.

555. Construct, as in Ex. 551, an equilateral triangle E equivalent to the given square. Then by Art. 364, construct an equilateral triangle X, such that $X : E = 7 : 5$.

EXERCISES, pp. 201–212.

556. An int. $\angle A$ of a regular polygon of n sides $= \frac{n-2}{n}$ st. $\angle$s;

$$\text{an int. } \angle A' \text{ of a regular polygon of } 2n \text{ sides} = \frac{2n-2}{2n} \text{ st. } \angle\text{s};$$

$$\therefore \angle A : \angle A' = n - 2 : n - 1.$$

557. Draw GR perpendicular to AB. Since $GA = GB$, $AR = BR$. Now $\overline{GR}^2 \approx \overline{GA}^2 - \overline{AR}^2$; $\therefore 4n\overline{GR}^2 \approx 4n\overline{GA}^2 - 4n\overline{AR}^2$. But $GA = \frac{1}{2} GH$, and $AR = \frac{1}{2} AB$; $\therefore n(4\overline{GA}^2 - 4\overline{AR}^2) \approx n\overline{GH}^2 - n\overline{AB}^2 \approx$ the sum of the squares of the sides of the outer polygon minus the sum of the squares of the inner polygon; $\therefore 4n\overline{GR}^2 \approx$ etc.

558. $\angle GAB$ is measured by $\frac{1}{2}$ arc AB, and arc AB is $\frac{1}{n}$th of the circumference; $\therefore \angle GAB$ is measured by $\frac{1}{2n}$th of a circumference; $\therefore \angle GAB = \frac{1}{n}$ straight angle.

559. It will pass through the mid points of OB, OR, OA, and AF.

560. It is easily shown that P', the figure formed, is similar to P; etc.

561. Let AO produced meet the circumference in C, and join OB. $\angle AOB$ is measured by arc AB, $\frac{1}{2n}$th of the circumference; $\therefore \angle BOC$, the supplement of $\angle AOC$, is measured by $\left(\frac{1}{2} - \frac{1}{2n}\right) = \frac{n-1}{2n}$ of a circumference. Hence C must fall at a vertex of the figure, since arc BC contains $n - 1$ of the $2n$ arcs.

562. Let $2n$ be the number of sides. Then any chord joining opposite vertices subtends n of the $2n$ arcs, or $\frac{1}{2}$ of a circumference; it is therefore a diameter.

563. Join OB. Then OAB is an equilateral triangle of which OP is the altitude on AB; $\therefore AB$ or $OA = 2AP$. Now $\overline{OP}^2 \approx \overline{OA}^2 - \overline{AP}^2 \approx 4\overline{AP}^2 - \overline{AP}^2 \approx 3\overline{AP}^2$; $\therefore OP : AP = \sqrt{3} : 1$.

564. Since PC joins the mid points of AB and AO, PC is $\parallel$ to BO, and $=$ to $\frac{1}{2} BO = DO$. Similarly PD is $\parallel$ and $=$ to CO; also $CO = DO$; $\therefore OCPD$ is a rhombus, unless $\angle O$ is a right angle; in that case, $OCPD$ is a square. AOB will be a right angle when the polygon is a square.

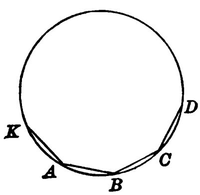
565. Let $ABCDEF$ be the regular hexagon inscribed in $\odot O$. Join ACE, forming an equilateral triangle. Through A, C, and E, draw tangents meeting in G, H, and K, so as to form the circumscribing equilateral $\triangle HKG$ (386). Each of the $\triangle$s HEC, EKA, AEC, $CAG \Bumpeq \frac{1}{4} \triangle HKG$ (Ex. 151), and the $\triangle$s DEC, FEA, BAC, are equal (69). Produce CB to meet AG in L. Then $CB = 2\,BL$ (Ex. 154); $\therefore \triangle BAC \Bumpeq \frac{2}{3} \triangle LAC \Bumpeq \frac{1}{3} \triangle CAG$; $\therefore ABCDEF \Bumpeq 2 \triangle CAG \Bumpeq \frac{1}{2} \triangle HKG$.

566. Let the equilateral $\triangle ABC$ be inscribed in $\odot O$. Draw radius ODE perpendicular to BC and join BO, BE. Then $BO = BE$ (408); $\therefore OD = \frac{1}{2} OE$.

567. Let $ABCDEF$ be the inscribed hexagon, and EAC the inscribed equilateral triangle. Draw OG perpendicular to AB. Join BO and produce; it will pass through E (Ex. 562). Then since $\angle EAB$ is a right angle (267), and OGB is also a right angle, OG, drawn through the mid point of BE, is parallel to AE; $\therefore OG = \frac{1}{2} AE$.

568. Let PA, PB be tangents drawn to the extremities of arc ACB, whose chord is AB. Arc $ACB > AB$ (389), and arc $ACB < PA + PB$ (390).

569. If n be the number of sides, and therefore of equal arcs, each angle is measured by $n - 2$ of these arcs; hence the polygon is equiangular.



570. Let ABC, BCD be any two successive angles of an equiangular polygon $ABC \cdots K$ of $2n + 1$ sides. Since $\angle ABC = \angle BCD$ (Hyp.), arc $ABC =$ arc BCD; $\therefore$ arc $AB =$ arc CD (Ax. 3); that is, any two alternate arcs are equal. Hence, calling arc AB the first arc, arc AB is equal to the 3d, 5th, $\cdots 2n + 1$th arc. But the $2n + 1$th arc is KA; $\therefore$ any two contiguous arcs are equal; hence all the sides are equal, *i.e.*, the polygon is equilateral.

571. Let $ABCD \cdots HK$ be a polygon of $2n+1$ sides, circumscribing $\odot O$ and touching it in $a, b, c, d \cdots h, k$. Each pair of tangents, Aa, Ak, drawn from A, one of the $2n + 1$ vertices, is equal; that is, $Aa = Ak$, $Bb = Ba$, $Cc = Cb, \cdots Kk = Kh$. Hence $Aa + Ba + Cc + Dc + \cdots Kk = Ak + Bb + Cb + Dd + Ed + \cdots Hh + Kh$. In each member of this equality are $2n + 1$ tangents, and they can be arranged in n pairs plus an odd tangent Kk in the first member and Ak in the second. Now each of those pairs forms one of the equal sides of the polygon; hence $Kk = Ak$; hence the tangents are all equal and the angles formed by them are equal.

572. Let ABC, BCD be successive angles of any polygon circumscribed about $\odot O$. Join OA, OB, OC. $\measuredangle$s OAB, OBA, OBC, OCB,

are equal (Ex. 200); $\therefore$ $\triangle AOB$, BOC are similar; $\therefore$ $AB:OB = BC:OB$; $\therefore$ $AB = BC$; etc.

573. Let $ABCDEF$ be the given inscribed hexagon, and $abcdef$ the figure formed by drawing chords as directed. In $\triangle aAF$, BAb, $\angle aFA = \angle aAF = \angle bBA = \angle bAB$, and $AF = AB$; $\therefore$ $aF = aA = bA = bB$. Also in $\triangle Aab$, $\angle A = 60°$; $\therefore$ $ab = AF = bB$. Similarly each side of $abcdef$ is one third of a side of the inscribed equilateral triangle, and its angles are equal, each being complement of an angle of an equilateral triangle; etc.

574. Join Oa, Ob. It is easily shown that $Oab = \frac{1}{6}\ abcdef = \frac{1}{18}$ of $ABCDEF$; $\therefore$ $abcdef : ABCDEF = 1:3$.

575. A semicircle is both a segment and a sector (400, 401).

576. $\odot\, O : \odot\, O' = R^2 : R'^2 = 2:1$; $\therefore$ $R:R' = \sqrt{2}:1$.

577. Let B, B' denote the segments; then $B:B' = 3^2 : 5^2 = 9:25$.

578. $B:B' = \overline{OA}^2 : \overline{O'A'}^2 = 5:3$; $\therefore$ $OA:O'A' = \sqrt{5}:\sqrt{3}$.

579. Let x denote the number of degrees in the arc of segment B; then segment $B : \odot\, O = x : 360 = 2:9$; $\therefore$ $x = 80$.

580. Since $\angle CAD = 38°$, arc $DC = 76°$; $\therefore$ arc $AC = 152°$; $\therefore$ $\angle AOC = 152°$.

581. Since $\angle CAF = 41°$, arc $AC = 82°$; $\therefore$ the angle formed by lines drawn from A and C to any point in arc $AC = \frac{1}{2}\,(360° - 82°) = 139°$.

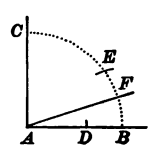
582. Arc $AE = \frac{1}{8}$ of a circumference $= 45°$; $\therefore$ arc $AHBE = 315°$; $\therefore$ $\angle APE = 157°\ 30'$.

583. Arc $AG = \frac{3}{8}$ of a circumference $= 135°$; $\therefore$ arc $ADBG = 225°$; $\therefore$ $\angle AQG$, formed by lines from A and G to any point Q in $ADBG$, $= 112°\ 30'$; $\therefore$ $\angle AQG : \angle APE = 5:7$.

QUESTIONS, p. 221.

584. By means of the straight line and the circle a circumference can be divided into 2, 3, 4, 5, 6, 8, 10, 12, or 15 equal arcs.

585. Let ABC be an equilateral triangle, and O the common center of the inscribed and circumscribed circles (374). Draw OD perpendicular to BC, produce to meet arc BC in E, and join BO, BE. Arc $BE = \frac{1}{2}$ arc $BC = \frac{1}{6}$ circumference; $\therefore$ $BE = BO$ (408); $\therefore$ $\angle BEO = \angle BOE$; $\therefore$ BD bisects OE; *i.e.*, the radius of the inscribed is equal to $\frac{1}{2}$ that of the circumscribed circle.



586. By means of Art. 409. Thus draw an arc subtending the given rt. $\angle BAC$, divide AB in medial section in D, lay off an arc BE with chord $= AD$, and join A with F, the mid point of arc BE. $BE = \frac{1}{5}$ of a semicircumference; $\therefore$ $BF = \frac{1}{5}$ of a quadrant.

587. Since 3 and 2 are both prime to 7 and less

than $\frac{7}{2}$, there are two different stellate heptagons (411). Since 4 and 2 are both prime to 9 and less than $\frac{9}{2}$, there are two different stellate nonagons. Since 5, 4, 3, 2, are each prime to 11 and less than $\frac{11}{2}$, there are four different stellate undecagons. Since 6, 5, 4, 3, 2, are each prime to 13 and less than $\frac{13}{2}$, there are 5 stellate 13-gons.

588. Let the sides AB, DC, of the regular pentagon $ABCDE$ be produced to meet in F. Since $\angle ABC$, an exterior angle of $\triangle FBC$, $= 108^\circ$, $\angle FBC = 72^\circ$; $\therefore \angle F = 36^\circ$.

589. The interior angles of an octagon $=(8-2)\times 180^\circ = 1080^\circ$; of a decagon $=(10-2)\times 180^\circ = 1440^\circ$; of a pentadecagon $=(15-2)\times 180^\circ = 2340^\circ$. Each interior angle of a regular polygon of n sides $= \frac{n-2}{n}$ straight angles $=\left(1-\frac{2}{n}\right)$ straight angles. Now if n becomes indefinitely great, $\frac{2}{n}$ becomes indefinitely small, and $1-\frac{2}{n}$ tends towards 1 as limit. Hence each interior angle tends towards a straight angle as limit. Since each exterior angle is the supplement of the adjacent interior angle, as the latter tends towards a straight angle as limit, the former tends towards 0° as limit.

590. $C = 10$ in. $\times \pi = 31.42$ in. $S = (5^2 \times \pi) = 78.54$ sq. in.

591. $C = 30$ in. $\times 2\pi = 188.50$ in. $S = (30^2 \times \pi) = 2827.44$ sq. in.

592. $\pi \cdot R^2 = 100$; $\therefore R = 10 \div \sqrt{\pi}$; $C = 2\pi \cdot R = 2\pi \times 10 \div \sqrt{\pi} = 20\sqrt{\pi} = 35.45$ in.

593. $\pi \cdot R^2 = 6$; $\therefore R = \sqrt{6 \div \pi} = 1.38$ ft.

594. $\pi \times 2R = 12$; $\therefore R = 12 \div 2\pi = 1.91$.

595. $\pi \cdot R'^2 = n\pi R^2$; $\therefore R' = \sqrt{n}R$.

596. $\pi \times 7^2 - \pi \times 5^2 = \pi \times 24 = 75.398$ sq. ft.

597. $\pi \cdot R^2 - \pi \cdot r^2 = \frac{2}{5}\pi \cdot R^2$; $\pi \cdot r^2 = \frac{3}{5}\pi \cdot R^2$; $\therefore r = \sqrt{\frac{3}{5}}\,R$.

598. $25 : x = 4^2 : 7^2$; $\therefore x = 76.5625$. *Ans.* 76.5625 sq. in.

599. $\pi \cdot 3^2 - (\sqrt{9+9})^2 = 9(\pi - 2) = 10.27$. *Ans.* 10.27 sq. in.

600. $4^2 - \pi \times 2^2 = 16 - 12.57 = 3.43$. *Ans.* 3.43 sq. in.

601. Since (415) $AC = R\sqrt{3} = 10\sqrt{3}$, $AC = 17.32$ in.

602. Since $10 = R\sqrt{3}$, $R = 10 \div \sqrt{3} = 5.77$ in.

603. Since (417) $AB = \frac{1}{2}R\sqrt{10 - 2\sqrt{5}} = 4\sqrt{5.528}$, $AB = 9.40$ in.

604. Since $9 = \frac{1}{2}R\sqrt{10-2\sqrt{5}}$, $R = 18 \div 2.35 = 7.66$ in.

605. Since (419) $AE = R\sqrt{2-\sqrt{2}} = 6 \times .765$, $AE = 4.59$ in.

606. Since $10 = R\sqrt{2-\sqrt{2}}$, $R = 10 \div .765 = 13.07$ in.

607. Since (420) $BF = \frac{1}{2}R(\sqrt{5}-1) = 5 \times 1.236$, $BF = 6.18$ in.

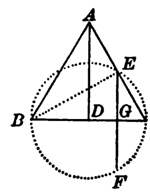
608. Since $3 = \frac{1}{2}R(\sqrt{5}-1)$, $R = 6 \div 1.236 = 4.85$ in.

THEOREMS, p. 222.

609. Let A and A' denote an angle of the polygons of n and $n+2$ sides, respectively. Then $\angle A : \angle A' = \dfrac{n-2}{n} : \dfrac{n}{n+2} = n^2 - 4 : n^2$.

610. For that point is equidistant from all the sides (101).

611. Let the regular hexagon $ABCDEF$ and the equilateral $\triangle ACE$ be inscribed in $\odot O$. Join OA, OC, OE. It is easily shown that BO is a parallelogram bisected by AC; etc.



612. Let AD be the altitude on BC of equilateral $\triangle ABC$. On BC as diameter describe circumference $BECF$, cutting AC in E. Through E draw $EF \parallel$ to AD, cutting DC in G, and join BE. Since BEC is a right angle, $AC = 2\,EC$; $\therefore AD = 2\,EG$ (148). But $EF = 2\,EG$ (172); $\therefore AD = EF$, and EF is a side of an inscribed triangle, since $\angle CEG = \angle CAD$; whence $CF = \frac{1}{6}$ circumference.

613. For (see preceding diagram) $BG = BD + DG = 3\,DG$, and $BD = DC = 2\,DG$; etc.

614. It is easily shown (Ex. 151) that the circumscribed equilateral triangle $E \Leftrightarrow 4\,E'$, the inscribed equilateral triangle. But H, the inscribed hexagon, $= 2\,E'$ (Ex. 611); $\therefore E = 2\,H$; $\therefore E : H = H : E'$.

615. In the diagram for Ex. 612, EC is the side of an inscribed hexagon, and EF that of an inscribed equilateral triangle. Now $\overline{EF}^2 = 4\,\overline{EG}^2$, and $\overline{EC}^2 = \overline{EG}^2 + \overline{CG}^2 = \overline{EG}^2 + \frac{1}{4}\overline{EC}^2$; $\therefore \overline{EC}^2 = \frac{4}{3}\overline{EG}^2$; $\therefore \overline{EF}^2 : \overline{EC}^2 = 4 : \frac{4}{3} = 3 : 1$.

616. Let ABC be the inscribed equilateral triangle. Join DE, the mid points of arcs AB, AC, and let DE cut AB, AC in F, G respectively. Since $DB = \text{arc}\, EC$ (Const.), DE is parallel to BC; $\therefore AFG$ is an equilateral triangle; $\therefore AF = AG = FG$. Now $\angle$s D, FAD, E, and EAG are equal, being measured by equal arcs; $\therefore FD = FA$, $GE = AG$; $\therefore DF = FG = GE$.

617. Let $ABCDE$ be a regular pentagon, and AC, AD diagonals drawn from any vertex A to the opposite vertices C, D. Circumscribe a circumference about $ABCDE$. Since $BC = CD = DE$, arc BC = arc CD = arc DE; $\therefore \angle BAC = \angle CAD = \angle DAE$.

618. We prove, as in the preceding exercise, that the angles formed by the diagonals are equal, as being measured by equal arcs; and there are $n-2$ of them, since they are subtended by all the n equal arcs, except the two whose chords form the divided angle A.

619. Let $ABCDE$ be the regular pentagon, and *abcde* the figure formed by the intercepts of the diagonals. The figure is equiangular,

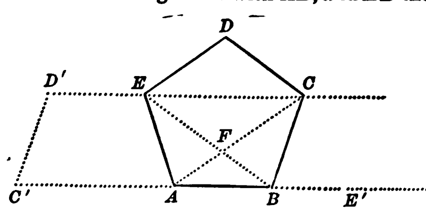
since its angles are measured by equal arcs; $\angle a$ for example, by one half the sum of the arcs of the circumscribing circle which are subtended by the chords AB, CD, DE. The figure is also equilateral, its sides being the bases of equal isosceles triangles.

620. Let the figure *abcde* in the diagram for the preceding exercise be the given pentagon. The isosceles triangles formed on each side by producing the alternate sides are easily proved equal, hence the lines AB, BC, etc., joining their vertices are easily proved equal, and the $\triangle ABC$, BCD, etc. are also equal.

621. In the diagram for Ex. 619, $\triangle aAB$, bBC are easily proved equal and isos., $\therefore aA = aB = bB = bC$. Again, $\angle BAa = \angle bBa$, and $\angle AbB$ is common to $\triangle AbB$ and Bab; $\therefore Ab : bB = bB : ab$; or $Ab : Aa = Aa : ab$; $\therefore Ab + Aa : Aa = Aa + ab : ab$; $\therefore Ab + bC : Aa = Ab : ab$; or $AC : Ab = Aa : ab = Ab : Aa$.

622. Circumscribe a circle about the pentagon $ABCDE$; then $\angle D = \angle CFE$, each being measured by equal arcs, *i.e.*, by $\frac{1}{2}$ the sum of three of the arcs subtended by the sides of the pentagon. Similarly $\angle DEF = \angle DCF$, each being measured by $\frac{1}{2}$ the sum of two of those arcs. Hence $FCDE$ is a parallelogram (141).

623. Let $ABCDE$ be a regular pentagon. Join AC, BE, intersecting in F. FD is a parallelogram (Ex. 622), and $EF = DC = EA$; $\therefore \angle EAF = \angle EFA =$ supplement of $\angle F$ or $\angle D$ or $\angle EAB$. Hence if trapezoid $ACDE$ be folded back, with AE as axis, AC will take the position AC' in the same straight line with AB, and ED the position ED'



parallel to $C'B$. In the same way trapezoid $BEDC$ may be folded back so as to have BE in the same line with BA, and CD in the same line with CD'. It is evident that the strip $D'E'$ may be folded back so as to coincide with the pentagon by drawing from E, any convenient point in one of its edges, EA, making $\angle EAC' = 72°$, making $AB = AE$, $\angle ABC = \angle BAE$; etc.

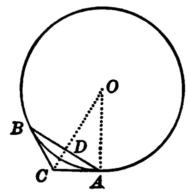
624. $P = \frac{1}{2} R\sqrt{10 - 2\sqrt{5}}$ (417); $H = R$ (418); $D = \frac{1}{2} R(\sqrt{5} - 1)$ (420); $\therefore P^2 = \frac{R^2}{4}(10 - 2\sqrt{5}) = R^2 + \frac{R^2}{4}(6 - 2\sqrt{5}) = H^2 + D^2$.

625. Let P be the point from which perpendiculars Pa, Pb, $Pc \cdots$

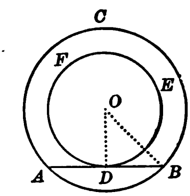
Pk, are drawn to the n equal sides of the polygon Q. Join P with the vertices, dividing Q into n triangles with bases each equal to a side.

$$\text{Then}\quad Q \simeq \tfrac{1}{2}(Pa\cdot AB + Pb\cdot AB + \cdots Pk\cdot AB)$$
$$\simeq \tfrac{1}{2}AB(Pa + Pb + \cdots Pk) \simeq \tfrac{1}{2}AB\cdot nOR \text{ (387)},$$

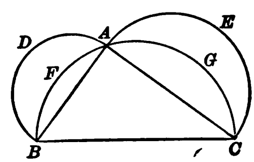
OR being the apothem; $\therefore\ Pa + Pb + \cdots Pk = n\cdot OR$.



626. Let AB be a side of a regular polygon P inscribed in $\odot\ O$. At A and B draw tangents meeting in C; draw OC cutting AB in D, and join OA. OA is the radius and OD the apothem of P, and OC is the radius of a polygon similar to P circumscribed about $\odot\ O$; OA is $\perp$ to AC (191), and OC is $\perp$ to AB (74); $\therefore$ since $\angle O$ is common to rt. $\triangle$s ODA and OAC, they are similar; $\therefore\ OD : OA = OA : OC$.



627. Let Ⓢ ABC, DEF have the common center O, and AB be a chord of ABC tangent to DEF at D. Join OB, OD. OD is $\perp$ to BD (191). Since $\odot ABC = \pi\cdot\overline{OB}^2$, and $\odot DEF = \pi\cdot\overline{OD}^2$, $\odot ABC - \odot DEF \simeq \pi(\overline{OB}^2 - \overline{OD}^2) \simeq \pi\overline{BD}^2$; and BD is the radius of a $\odot$ whose diameter is AB.



628. Let ABC, DAB, and EAC be semicircles described upon the hypotenuse BC and the arms AB, AC of rt. $\triangle ABC$. Now $\tfrac{1}{2}\odot ABC = \pi\cdot\tfrac{1}{8}\overline{BC}^2$, $\tfrac{1}{2}\odot DAB = \pi\cdot\tfrac{1}{8}\overline{AB}^2$, and $\tfrac{1}{2}\odot EAC = \pi\cdot\tfrac{1}{8}\overline{AC}^2$ (398);

$$\therefore\ \tfrac{1}{2}\odot ABC - (\tfrac{1}{2}\odot DAB + \tfrac{1}{2}\odot EAC)$$
$$= \pi\,\tfrac{1}{8}(\overline{BC}^2 - \overline{AB}^2 - \overline{AC}^2) = 0;$$

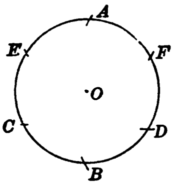
$\therefore\ \tfrac{1}{2}\odot DAB + \tfrac{1}{2}\odot EAC \simeq \tfrac{1}{2}\odot ABC$. From each of these equals take the sum of the figures FAB, GAC; then $ADBF + AECG \simeq \triangle ABC$.

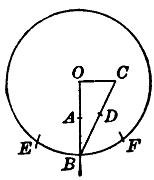
REMARK. — If the triangle is isosceles, $2\,ADBF \simeq$ isosceles right triangle $ABC \simeq \overline{AB}^2$. These figures, known as the *lunes* of Hipparchus, are interesting as being probably the first curvilinear figures whose quadrature was effected.

PROBLEMS, p. 224.

The first eight of these problems are easily solved by means of the constructions given in Arts. 406–409, and in Art. 385. Yet as the inscribed or circumscribed trigon, pentagon, and hexagon are so frequently required in demonstration, it may be useful to suggest the easiest way of producing them.

To construct an equilateral triangle inscribed in a circle, with any convenient radius OA describe a circumference. If a point A on this circumference is to be a vertex, mark the extremities A, B of the diameter from A; then with B as center, and radius BO, describe an arc cutting the circumference in C and D. A, C, D, are the required vertices. If an inscribed hexagon is required, from both B and A as centers, with radius BO, describe arcs cutting the circumference in C, D, E, and F. Then A, E, C, B, D, F, are the required vertices.





To construct a pentagon inscribed in a circle, from any center O draw a line perpendicular to the direction in which the base is to lie. Lay off any convenient equal lengths OA, AB, and describe a circumference with radius OB. Draw OC perpendicular to OA and equal to OA; join CB, and lay off CD equal to OC. From B as center, with radius BD, describe an arc cutting the circumference in E and F. E, F are vertices of the required pentagon.

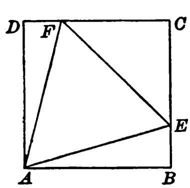
To obtain these polygons circumscribed, we first find the vertical points as above, then proceed by Art. 385.

629. See Art. 407. **631.** See Art. 406. **633.** See Arts. 407 and 385.

630. See Art. 409. **632.** See Art. 406. **634.** See Arts. 409 and 385.

635. See Arts. 409 and 385. **636.** See Arts. 409 and 385.

637. Let $ABCD$ be the given rectangle. At E, the mid point of AB, erect EF perpendicular to AB and equal to AE. Produce FE to meet CD in G, and produce farther to H so that $GH = GD$. Join FA, FB, and draw HDK, HCL to meet FA, FB produced in K and L respectively. $FLHK$ is easily shown to be a square.



638. Let $ABCD$ be the given square, and AEF the required equilateral triangle with a vertex in A. Since $AB = AD$, and $AE = AF$, rt. $\triangle ABE =$ rt. $\triangle ADF$; $\therefore \angle BAE = \angle DAF$. But $\angle BAE + \angle DAF$ or $2\angle BAE = 90° - 60° = 30°$; $\therefore \angle BAE = 15°$. Hence trisect $\angle BAD$ (Ex. 246); then bisect the two outer parts by AE, AF, and join EF; etc.

639. Let AB be a side of the given equilateral triangle. Find a line CD such that $CD : AB = 1 : \sqrt{2}$ (363); then the equilateral triangle constructed upon CD as base is the triangle required (342).

640. Let AB be a side of the given square. Find a line CD such that

$CD : AB = \sqrt{2} : \sqrt{3}$ (363); the square constructed upon CD as base is the required square.

641. Let AB be a side of the given pentagon. Find a line CD such that $CD : AB = \sqrt{3} : \sqrt{4}$; the pentagon constructed upon CD as base is the required pentagon.

642. Let AB be a side of the given hexagon. Find a line CD such that $CD : AB = \sqrt{4} : \sqrt{5}$; the hexagon constructed upon CD as base is the required hexagon.

643. Let AB be the radius of the given circle. Find a line CD such that $CD : AB = \sqrt{5} : \sqrt{6}$; the circle described with radius equal to CD is the circle required.

NOTE.—For solutions to exercises in the Appendix to Plane Geometry bound separately, see p. 105 of this Key.

PART II.

EXERCISES, pp. 227–257.

644. The revolving plane would coincide with MN on coming to contain the point B. If the plane continued to revolve, it would, after a semi-revolution, again coincide with MN.

645. Such an intersection must be either a broken line or a curve. The intersection of a plane with the surface of an apple, for example, is a curve.

REMARK.—Illustrate by cutting an apple, or similar object, in two.

646. The converse of Cor. 4, Prop. II., viz., *A straight line is the intersection of two planes*, is not generally true. Thus the intersection of a plane with any surface on which a straight line can be drawn, such as a cylindrical or conical surface, or the intersection of two such surfaces, may be a straight line.

647. The figure is not a stellate polygon, because the chords do not form a continuous broken line, but two separate triangles.

648. As these planes have the points A and P in common, the line joining these points must lie in each, and be their common intersection.

649. Since AB, AC, AD, are the hypotenuses of right triangles having the arm AP common, the circumferences described must pass through the points A and P, and have AP as a common chord.

650. $\overline{AD}^2 \approx \overline{AP}^2 + \overline{PD}^2 \approx \overline{AP}^2 + \overline{AB}^2$; $\therefore\ \overline{AD}^2 - \overline{AP}^2 = \overline{AB}^2$;

$$\therefore\ (AD + AP)(AD - AP) = \overline{AB}^2.$$

651. If planes be passed through DD' and AA', and through DD'

and CC', DA', DC' will be par'ms; hence DA, DC are $\parallel$ and $=$ to $D'A'$, $D'C'$ respectively; hence $\angle D = \angle D'$, and $\triangle DAC = \triangle D'A'C'$.

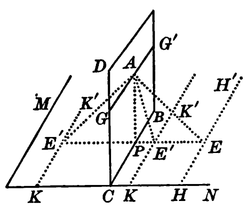
652. In the diagram for Prop. XIII., suppose the planes AB', AC', BC', are $\perp$ to MN, the plane of $\triangle ABC$. Since AA', BB', CC', are then $\perp$ to MN (475), the $\angle$s BAC, ABC, ACB, are the plane angles of the dihedral angles whose edges are AA', BB', CC', respectively; hence no two of these dihedral angles are equal, two are equal, or all are equal according as $\triangle ABC$ is scalene, isosceles, or equilateral.

653. Since $FG : AC = DF : DA$, and $FE : BD = AF : DA$, we have

$$FG \cdot DA = AC \cdot DF, \text{ and } FE \cdot DA = BD \cdot AF;$$

$$\therefore FG : FE = AC \cdot DF : BD \cdot AF.$$

654. Let HH', KK' be the intersections with plane MN of the planes passed through GG'. Since GG' is parallel to BC (Hyp.), GG' is parallel to plane MN (448); $\therefore HH'$, KK' are parallel to GG' (449); $\therefore HH'$ is parallel to KK'.



655. In the diagram for Ex. 654, suppose a plane $AEPE'$ perpendicular to MN, cutting GG', BC, HH', KK', in A, P, E, E', respectively. The plane is perpendicular to BC, HH', KK'. Join AE, AE'; then, according as PE is or is not equal to PE', rt. $\triangle APE$ is or is not equal to rt. $\triangle APE'$, and $\angle PEA$ is or is not equal to $\angle PE'A$; *i.e.*, the dihedral angles measured by PEA, $PE'A$ are or are not equal.

656. As shown in the preceding exercise, if BC is equidistant from HH' and KK', rt. $\triangle APE =$ rt. $\triangle APE'$, and $\angle PAE = \angle PAE'$.

657. KK' being on the same side of BC as HH', let plane APE cut KK' in E', and join AE'; then $\angle AEE' = \angle AE'P - \angle EAE'$, and these angles are the plane angles that measure the dihedral angles whose edges are HH', KK', and GG', respectively.

658. In the diagram for Prop. XX., if plane PQ is perpendicular to plane RS, BQ is perpendicular to SB, and therefore to plane RS. Similarly BS is perpendicular to plane PQ, while AB is perpendicular to plane MN; hence AB, BQ, BS, are perpendicular to each other.

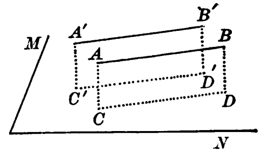
659. Let the production of QB be BQ'. If $\angle SBQ = 140°$, then $\angle SBQ' = 180° - 140° = 40°$; $\therefore$ dihedral $\angle SABQ$: dihedral $\angle SABQ'$ $= 140 : 40 = 7 : 2$.

660. In the plane of PE and PF, quadrilateral $PEOF$ has $\angle$s PEO, PFO right angles; $\therefore$ points P, E, O, F, are concyclic (Ex. 398).

661. If FO is equal to FP, PFO is an isosceles right triangle; $\therefore$ $\angle FOP = 45°$; similarly $\angle EOP = 45°$; $\therefore \angle EOF = 90°$; $\therefore$ $CABD$ is a right dihedral angle.

662. Suppose FE joined. If $FE = FP = EP$, PEF is an equilateral triangle, $\angle EPF = 60°$, and $\angle EOF = 180° - 60° = 120°$.

663. In the diagram for Prop. XXII., draw AB' parallel to CD. Then CB' is a parallelogram, and $AB' = CD$. But $AB = CD$ (Hyp.); $\therefore$ $AB = AB'$, which cannot be the case unless AB coincides with AB and is therefore parallel to CD.



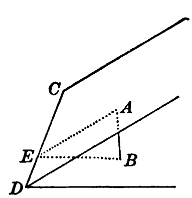
664. Let AB, $A'B'$ be parallel lines having projections CD, $C'D'$ upon the same plane MN. (1) If AB, $A'B'$ lie in the same plane $\perp$ to MN, then CD, $C'D'$ coincide. (2) If AB, $A'B$ do not lie in the same plane $\perp$ to MN, CD, $C'D'$ will not coincide, but will be parallel. For otherwise they would meet in some point O, if produced, and from O could be drawn, in their respective planes of projection, lines OE, OE' $\parallel$ to AB, $A'B'$ respectively. OE' being $\parallel$ to $A'B'$ would also be $\parallel$ to AB (445), so that through the same point would be drawn two parallels OE, OE', to the same line AB, which is impossible.

665. For the point in which BA produced meets MN, is its own projection, and lies, therefore, in DC produced (481).

666. BA being produced to meet DC produced in E (Ex. 665), we have $ED : EC = BD : AC = m : n$; $\therefore$ $ED - EC : EC = m - n : n$; *i.e.*, $CD : EC = m - n : n$.

667. (1) All lines in the same plane of projection have their projections in the same straight line, irrespective of what angles they make with their common projection. (2) All lines in parallel planes of projection have their projections parallel (Ex. 664), irrespective of what angles they make with their respective projections.

668. Since the line with which a line oblique to a plane makes the least angle is that part of its projection with which it makes an acute angle (482), the line with which it makes the greatest angle is that part of its projection with which it makes an obtuse angle. For this obtuse angle is the supplement of the least angle the oblique line makes with any line in the plane.



669. Let AB be the line making an angle of $42°$ with each of the planes PQ, MN, intersecting in CD. From A draw $AE \perp$ to CD, and pass a plane through AB and AE; the plane AEB is $\perp$ to CD and therefore to PQ and MN (470). Hence its intersections with MN and PQ are the projections of AB upon those planes. Now $\triangle EAB$, having two angles each equal to $42°$, has a vertical $\angle AEB$ equal to $96°$, which is the plane angle of dihedral angle CD (462).

670. There are four trihedral angles formed by PQ and RS with MN, and having the common vertex B.

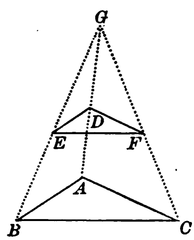
671. If SBQ is a right angle, then since ABQ, ABS are each right angles (475), the trihedral $\angle B\text{-}ASQ$ is trirectangular; and similarly of the other trihedrals.

672. The intersection of a plane with any two edges of the polyhedral angle, forms the base of a triangle and cannot be $\perp$ to both (84). A plane passed $\perp$ to the edge of the dihedral angle formed by any two faces of the polyhedral angle, produced if necessary, will be $\perp$ to each of these faces; and, as seen above, a plane can be $\perp$ to only one edge.

673. Let the lines be A, B, and C. Planes can be passed through, and determined by, A and B, A and C, B and C; *i.e.*, three planes.

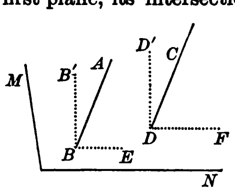
674. Let the lines be A, B, C, and D. They can be combined, two and two, in six ways: A and B, A and C, A and D, B and C, B and D, C and D. Hence six planes can be determined by these lines.

675. Let the line AB be parallel to each of the planes MN, PQ, intersecting in CD. Through AB pass a plane $\parallel$ to MN and intersecting PQ in EF. Then AB is $\parallel$ to EF (449), and CD is $\parallel$ to EF (451); $\therefore$ AB is $\parallel$ to CD.



676. Let ABC, DEF be two similar triangles, having AB, AC, BC, homologous and $\parallel$ to DE, DF, EF, respectively, but $EF < BC$. Join AD, BE, CF. Since $EF < BC$, BE is not $\parallel$ to CF; $\therefore$ BE, CF will meet, if produced, in a point G. Join GD. Since EF is $\parallel$ to BC, $GB:GE = BC:EF$. But $\triangle$s ABC, DEF being similar, $GB:GE = BA:ED$; $\therefore$ $GE:ED = GB:BA$. Since ED is $\parallel$ to BA, $\angle GED = \angle GBA$; $\therefore$ $\triangle GED$ is similar to $\triangle GBA$; $\therefore$ $\angle GDE = \angle GAB$; $\therefore$ $\angle GDE$ is supplement of $\angle ADE$; $\therefore$ DA is in the same straight line with DG (49).

677. For if a plane be passed through the line perpendicular to the first plane, its intersections with all the planes will be parallel, since they make equal angles with the same line.



678. Let the $\parallel$'s AB, CD meet plane MN in B and D respectively. Through AB, CD pass planes $\perp$ to MN, and intersecting it in BE, DF respectively. In these planes erect perpendiculars BB', DD', to BE, DF. Then $\angle ABB' = \angle CDD'$, since AB is $\parallel$ to CD, and BB' is $\parallel$ to DD' (443); $\therefore$ $\angle ABE = \angle CDF$ (44).

679. The points lie in the plane that is perpendicular to the given plane and intersects it in the given line.

QUESTIONS, p. 258.

680. One point determines only a point; two, a straight line; three, a plane.

681. To those contained in Props. I. and V.

682. The required locus is the line perpendicular to the plane of the circumference through its center. This may be proved by joining any point in the perpendicular with any two points in the circumference, and these points with the center.

683. The required locus is the line perpendicular to the plane of the triangle through the intersection of the bisectors of any two of its angles. This may be proved by joining any point in the perpendicular with the vertices of those angles; etc.

634. The required locus is the line perpendicular to the plane of the triangle, through the intersection of the perpendiculars drawn to any two of its sides at their respective mid points. This may be proved by joining any point in the perpendicular with those points; etc.

685. To that contained in Prop. XXIII.

686. The required locus is the plane that is perpendicular to the given line at the given point (434).

687. To those contained in Props. XXVI. and XXVII. respectively.

683. The locus is the plane perpendicular to the given plane and intersecting it in the given line. This follows from Arts. 479, 480, 481.

689. The required locus consists of two planes parallel to the given plane, one on each side of it and at the given distance from it.

690. To Axiom 10 and Prop. XXXII. respectively.

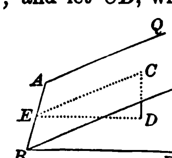
691. To that contained in Prop. VIII.

692. To that contained in Prop. V.

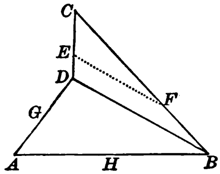
693. (1) The locus is the plane passed through the intersection of the given planes and bisecting their dihedral angle (478). (2) The locus is the plane perpendicular to the mid point of any line intercepted between the given planes and perpendicular to them (450).

THEOREMS, p. 258.

694. Let the planes AP, BQ intersect in AB; and let CD, which is $\perp$ to AP at D, meet BQ in C. Draw DE $\perp$ to AB, and join CE. Since plane CDE is $\perp$ to AB at E, it is also $\perp$ to plane BQ (470); hence EC is the projection of CD on BQ.



695. Let $ABCD$ be a quadrilateral in space; *i.e.*, CDB makes with ADB a dihedral angle whose edge is DB. If a plane be passed parallel to ADB and

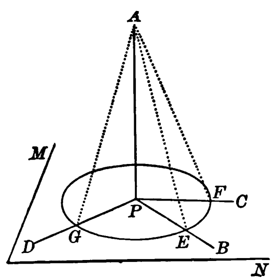


cutting CDB in EF, EF will be parallel to DB (451); hence $CE : ED = CF : FB$.

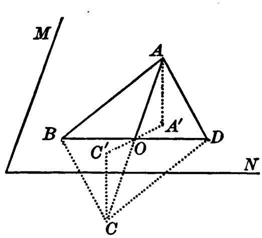
966. Let $ABCD$, as in Ex. 695, be a quadrilateral in space, and E, F, G, H, the mid points of CD, CB, AD, AB, respectively. In the plane passed through EF, GH, suppose EG, FH joined. Then since EF, GH are parallel to DB and equal to $\frac{1}{2}$ DB, EF is parallel and equal to GH; $\therefore$ $EGFH$ is a parallelogram (142).

697. Let AB, CD, EF, be any three of the parallel intersections, and P be the given point. Through P draw QR $\parallel$ to those intersections, and through P pass a plane $\perp$ to QR. This plane will be $\perp$ to AB, CD, EF, etc., since these lines are $\perp$ to it (444), and therefore to the planes of which these are the intersections (470); hence all the perpendiculars from P to these planes will lie in that plane.

698. In the diagram for Ex. 694, suppose CD equally inclined to AP and BQ. Through CD pass a plane $\perp$ to AB and cutting it in E. This plane, being $\perp$ to both AP and BQ (470), contains the projections CE, DE of CD upon BQ, AP resp. Now $\angle ECD = \angle EDC$ (Hyp.); $\therefore$ $ED = EC$.



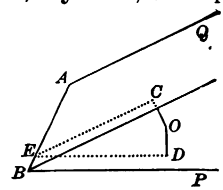
699. A being without the plane, let AP make equal angles with the lines PB, PC, PD, in plane MN. With P as center, describe a circumference cutting PB, PC, PD, in E, F, G, respectively, and join AE, AF, AG. Since the angles at P are equal (Hyp.), $AE=AF=AG$ (66, 70). Now the locus of all points at a distance AE from A is a circumference whose center is the foot of the perpendicular from A, and P is the center of the only circumference that can pass through E, F, and G (185); hence AP is $\perp$ to MN.



700. Let the plane MN intersect the parallelogram $ABCD$ in the diagonal BD, and let AA', CC' be perpendiculars from A and C to plane MN. Join A' and C' with O, the intersection of diagonal AC with BD. AA' is $\parallel$ to CC' (443); $\therefore$ $\angle OAA' = \angle OCC'$ (112); also hypot. $OA =$ hypot. OA'; $\therefore$ $AA' = CC'$ (73, 70).

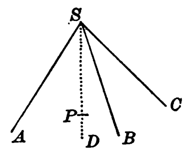
701. Let AP, BQ be planes intersecting in AB, and from O, a point between them, draw OD, OC $\perp$ to AP, BQ respectively.

Through OC, OD, pass a plane intersecting AP, BQ in DE, CE respectively. This plane is $\perp$ to AB (476); $\therefore DEC$ is the plane angle of dihedral $\angle QABP$, and is the supplement of $\angle O$, since in quadrilateral $OCED$, $\angle$s D and C are right angles. Hence $\angle O$ is the equal (44) of the plane angle of the dihedral angle formed by producing AP through AB.

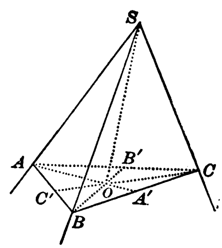


702. For the plane $\perp$ to the common intersection at any point will contain all the $\perp$'s to the intersection at that point (434).

703. Let SA, SB, SC, be the edges of trihedral $\angle S\text{-}ABC$. Through SA, SB pass planes bisecting the dihedral $\angle$s SA, SB respectively, and let these planes intersect in the straight line SD. Take any point P in SD. Since P is in the bisecting plane of dihedral $\angle SA$, P is equidistant from the faces ASB, ASC. For a similar reason P is equidistant from the faces BSC and ASB. Hence, being equidistant from ASC and BSC, P must lie in the bisecting plane of dihedral $\angle SC$; that is, any point in SD lies in the three bisecting planes.



704. On the edges of the trihedral $\angle S\text{-}ABC$ lay off SA, SB, SC, and through A, B, C, pass a plane forming, by its intersection with the faces, the $\triangle ABC$. Draw $AA' \perp$ to BC, and pass a plane through SA, AA'; this plane is $\perp$ to SBC (470). Similarly planes passed through SB, BB' and SC, CC' are $\perp$ to SAC, SAB respectively, BB', CC' being $\perp$ to AC, AB respectively. But AA', BB', CC', intersect in a point O (Ex. 152); $\therefore S$ and O being common to the three planes, the line joining SO is their common intersection.



705. In the diagram for Ex. 704, suppose $SA = SB = SC$; then the $\triangle$s SAB, SAC, SBC, are isosceles, and the bisectors of the face angles are also the medians of those triangles. Hence the planes passed through SA, SB, SC, and the bisectors of the opposite face angles, intersect $\triangle ABC$ in the medians to BC, AC, AB. But these medians intersect in a point O (Ex. 154); $\therefore S$ and O being common to the three planes, the line joining SO is their common intersection.

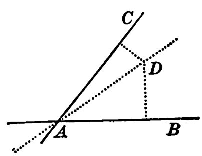
706. In the diagram for Ex. 705, since $\triangle$s SAB, SAC, SBC, are isosceles, the bisectors of the face angles are $\perp$ to BC, AC, AB, at their respective mid points. Hence the planes passed through the bisectors $\perp$ to the faces are also $\perp$ to BC, AC, AB, and their inter-

sections with $\triangle ABC$ form the perpendiculars at the mid points of its sides. But these perpendiculars meet in a point O (155); $\therefore$ S and O being common to the three planes, SO is their common intersection.

LOCI, p. 259.

707. The plane passed perpendicular to the line joining the given points A, B, at the mid point of AB, is the required locus; for the line joining that mid point C with any other point in the plane is perpendicular to AB; hence that point is equidistant from A and B (96).

708. The two planes passed perpendicular to the plane of the given lines, through the bisectors of the angles formed by those lines, constitute the required locus. For the foot of the perpendicular from any point in either of those planes to the bisector through which it passes, is equidistant from the given lines (101); hence that point is also equidistant from the given lines.

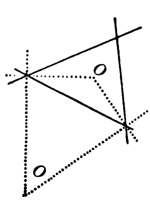


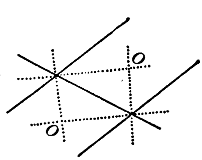
709. (1) Let the planes intersect. Pass a plane BAC perpendicular to the intersection of the given planes; then find (Ex. 457) the lines, such as AD, that constitute the locus of a point whose distances from AB, AC are in the given ratio. The planes passed through the intersection of the given planes and AD constitute the required locus. For the distances of any point D in either of those planes from the given planes are in the given ratio. (2) If the lines are parallel, we proceed similarly by means of Ex. 457.

710. Draw perpendiculars to any two sides of the triangle at their mid points, and let them intersect in O. The perpendicular to the plane of the triangle through O is the required locus; for O being equidistant from the vertices of the triangle, any point in the perpendicular to the plane of the triangle through O is equidistant from the vertices.

711. The perpendicular to the plane of the given triangle through the intersection of any two of the angles is the locus required (154).

712. The perpendicular to the plane of the quadrilateral through the center of the circumference that can be described about it (Ex. 398), is the required locus. For that center is equidistant from the vertices, which are concyclic.





713. The perpendicular to the plane of the circle through its center is the required locus.

714. If the given planes are parallel, it is evident that no point can be equidistant from all three. But if one plane intersect the other

two, the required locus will consist of the intersections of the planes that bisect the dihedral angles formed by the one plane with the other two. For any point O in either intersection will be equidistant from the three given planes (478).

715. Let S–ABC be the trihedral angle. The required locus is the intersection of the planes passed perpendicular to any two of the faces through the bisectors of their respective face angles. For any point in that intersection is equidistant from all the edges (Ex. 706).

716. (1) If the given planes intersect, the planes bisecting their dihedral angles constitute the locus of points equidistant from those planes. The plane perpendicular at the mid point of the line joining the given points, is the locus of points equidistant from the given points. The line or lines formed by the intersection of these two loci constitute the required locus. (2) If the given planes are parallel, the required locus will consist of the intersection of the plane equidistant from the given planes with the plane that is the locus of points equidistant from the given points.

PROBLEMS, p. 260.

717. Through any point C in the given line AB lying in plane MN, pass a plane PQ perpendicular to AB, and intersecting MN in CD. In PQ draw a line CE, making with CD an $\angle ECD$ equal to the given angle. The plane passed through AB, CE is the plane required.

718. Through any point C in the given line AB pass a plane $PQ \perp$ to AB, and intersecting the given plane MN in DE. In PQ draw a line CF, making with DE an $\angle CFE$ equal to the given angle; etc.

719. Pass a plane R perpendicular to the given edge at any point A; R will be perpendicular also to the faces P, Q of the given dihedral angle (470). Bisect the plane angle formed by the intersection of R with P and Q. The plane passed through the bisector and the given edge will be the plane required.

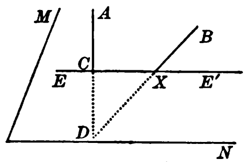
720. From the given point A draw $AB \perp$ to P, the given plane, and through A pass a plane $Q \perp$ to AB. This plane Q is $\parallel$ to plane P.

721. At any point A in the given plane P, draw AB perpendicular to P. On AB lay off AC equal to the given distance, and through C pass a plane Q perpendicular to AB; etc.

722. Through AB, the given line, pass two planes, one of which passes through C, the given point. In the plane containing C, draw CD perpendicular to AB, and in the other plane draw DE perpendicular to AB. The plane of CD and DE is the plane required.

723. Let S–ABC be the trihedral angle. Through the bisectors of two of the face angles, pass planes $\perp$ to those faces; the plane passed

through the vertex $S \perp$ to the intersection SO is the plane required. For through any point O pass a plane $\perp$ to SO, so as to intersect the faces in AB, AC, BC. Join OA, OB, OC. The rt. $\triangle$s SOA, SOB, SOC, are equal; $\therefore$ $\angle$s ASO, BSO, CSO, are equal; $\therefore$ the $\angle$s made by SA, SB, SC, with the $\perp$ plane through S are equal (44).



724. (1) Let the given points A, B be on the same side of the given plane MN. Through the points A and B pass a plane $P \perp$ to MN, and intersecting it in EE'. In plane P draw $AC \perp$ to EE', and produce to D, so that $CD = AC$. Draw BD, cutting EE' in X. X is the required point (Ex. 248).

(2) If A and B are on opposite sides of the plane, we proceed by a construction similar to that given above (Ex. 247).

725. Let A, B, C, be the given points, and MN the given plane. Find the locus of points equidistant from A, B, and C (Ex. 710); the point X, in which this locus meets MN, is the required point. If the plane of ABC is perpendicular to the given plane, there can be no point such as is required.

726. Find the locus of points equidistant from the given points. The point in which this plane cuts the given line is the required point. If the plane is $\parallel$ to the given line, there can be no point such as is required.

EXERCISES, pp. 263-280.

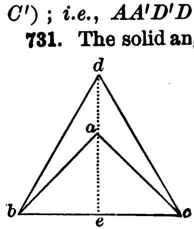
727. The sum of the lateral dihedral angles $= (5 - 2)180° = 540°$.

728. The prism MN has $5 + 2 = 7$ faces; $4 \times 5 + 2 \times 5 = 30$ face angles; $5 + 5 \times 2 = 15$ dihedral angles; $5 \times 2 = 10$ trihedral angles.

729. If the base of MN has n sides, the prism has $(n + 2)$ faces; $(4n + 2n) = 6n$ face $\angle$s; $(n + 2n) = 3n$ dihedral $\angle$s, and $2n$ trihedral $\angle$s.

730. Since $AC' \approx Ac'$ (525), $AC' - (ABCd - C') \approx Ac' - (ABCd - C')$; *i.e.*, $AA'D'D - d' \approx BB'C'C - c'$.

731. The solid angle whose vertex is A is a tetrahedral angle, having as faces $A'Aa$, $A'AD$, DAd, and dAa. The angle whose vertex is D is trihedral.



732. Let abc, dbc be respectively an isosceles right triangle, and an equilateral triangle, having the hypotenuse bc of abc as common base. Join da, and produce to meet bc in e; de is perpendicular to bc at its mid point (74). The triangles, having the same base, are as their altitudes. Now $ec : de = 1 : \sqrt{3}$, and $ec : ae = 1 : 1$; $\therefore$ $de : ae = \sqrt{3} : 1$. The prisms, having equal altitudes, are as their bases; $\therefore$ prism A : prism $B = 1 : \sqrt{3}$.

733. The perimeter of $\triangle abc = 2\,ab + bc = 2(ab + be) = 2\,(be \times \sqrt{2} + be) = 2\,be(\sqrt{2} + 1)$. The perimeter of $\triangle dab = 3\,bc = 6\,be$. The prisms having equal altitudes, their lateral areas are as the perimeters of their bases.

$\therefore$ lat. area A : lat. area $B = 2\,be(\sqrt{2} + 1) : 6\,be = \sqrt{2} + 1 : 3$.

734. In the diagram for Prop. XII., suppose PF joined. Since ABD is a regular polygon, the mid points of its sides are equally distant from P. It is thence easily shown that the inclination of each face, SAB, SBC, etc., is the same as that of SAE, which is measured by $\angle SFP$. Hence they are equally inclined to any plane passed perpendicular to SP, and therefore parallel to ABD.

735. (1) The face angles of the polyhedral angle must be equal. (2) The plane must cut off equal intercepts on edges of the polyhedral angle, so that the faces thus cut off shall be equal isosceles triangles, and the base, therefore, a regular polygon.

736. Let $a'b'c'd'e'$ be the perimeter of the section cut off by a plane passing through p', the mid point of pP, parallel to the base. Join ap, $a'p'$, AP, in the plane SPA. Since $p'P = \frac{1}{2}pP$, $a'A = \frac{1}{2}aA$ (149), also $a'b'$ is parallel to AB; $\therefore a'b' = \frac{1}{2}(ab + AB)$ (150). Similarly, $b'c' = \frac{1}{2}(bc + BC)$, etc.; $\therefore a'b'c'd'e' = \frac{1}{2}(abcde + ABCDE)$.

737. If $PF = PS$, SPF is an isosceles right triangle; $\therefore \angle SFP = 45°$, and is the plane angle of dihedral angle AE, since SF, PF are each perpendicular to the edge AE (75).

738. If $\angle SAB = 70°$, then $\angle ASB = 180° - 140° = 40°$; $\therefore$ the sum of the face angles having the common vertex S is $40° \times 5 = 200°$.

739. Since SPF is a right angle, if M be the mid point of SF, then $MP = MS$ (Ex. 144).

740. In the diagram for Prop. XII., let S–ABD be a regular pyramid of n sides. Since $\overline{SA}^2 \backsimeq \overline{SF}^2 + \overline{AF}^2 \backsimeq \overline{SF}^2 + \frac{1}{4}\,\overline{AE}^2$, $n\overline{SA}^2 \backsimeq n\overline{SF}^2 + \frac{n}{4}\,\overline{AE}^2$.

741. In the same diagram, PF being joined, since SPF is a right triangle of which SF is the hypotenuse, the perpendicular PQ from P to SF is a mean proportional between QS and QF (297).

742. In the same diagram, PF being joined, $\overline{PF}^2 \backsimeq \overline{SF}^2 - \overline{SP}^2$. Hence PF, the apothem, is one arm of a right triangle, of which SF is hypotenuse, and SP the other arm.

QUESTIONS, p. 290.

743. The least number of points that can limit a line is two; the least number of straight lines that can limit a surface is three; the least number of plane surfaces that can limit a solid is four.

744. The lateral area $= (25 \times 42)$ sq. in. $= 1050$ sq. in.

745. The volume $= (\frac{40}{12} \times \frac{7}{2})$ cu. ft. $= 11\frac{2}{3}$ cu. ft.

746. To that contained in Prop. III.

747. To that contained in Prop. XXXVI. Cor. 3, Book I., and Prop. IV., Book V., respectively.

748. Prop. VI. is analogous to Prop. III.; Prop. VII. and Prop. VIII., to Cor. 2 of Prop. III.; and Prop. IX., to Prop. II.

749. The lateral area $= 16\,(22 + 30 + 22 + 30)$ sq. in. $= 1654$ sq in. The basal areas $= 2\,(22 \times 30)$ sq. in. $= 1320$ sq. in. Hence the whole area $= 2974$ sq. in.

750. The volume $= (16 \times 22 \times 30)$ cu. in. $= 6\frac{1}{9}$ cu. ft.

751. Let P, P' denote the solids; then $P : P' = 2 \times 5 \times 14 : 3 \times 4 \times 10 = 7 : 6$.

752. The area of the base $= (96 \div 8\frac{1}{4})$ sq. ft. $= 11\frac{7}{11}$ sq. ft.

753. The altitude $= (120 \div 4\frac{5}{12} \times 6)$ ft. $= 4\frac{28}{53}$ ft.

754. The volume $= (4\frac{1}{2} \times 4\frac{1}{2} \times 4\frac{1}{2})$ cu. in. $= 91\frac{1}{8}$ cu. in. The entire surface $= (4\frac{1}{2} \times 4\frac{1}{2})$ sq. in. $\times\, 6 = 121\frac{1}{2}$ sq. in.

755. A gallon dry measure contains 268.8 cu. in., and $\sqrt[3]{268.8} = 6.45$. Hence the required box should have each edge equal to 6.45 in.

756. Let x denote the number of inches in the required edge. Since, by the conditions, $6x^2 = 144$, $x = \sqrt{24} = 4.9$ in.

757. The volume of the second prism $= (5\frac{1}{2} \times 2\frac{3}{4}) = \frac{121}{8}$ cu. in. Hence the altitude of the first prism should be $(\frac{121}{8} \div 3\frac{1}{2}) = 4\frac{9}{28}$ in.

758. Let B and S denote respectively the base and the required section. Then (546) $S : B = (6 - 2)^2 : 6^2 = 4 : 9$. Hence $S = \frac{4}{9} B = (12 \times \frac{4}{9})$ sq. ft. $= 5\frac{1}{3}$ sq. ft.

759. The lateral area $= (15 \times 7 \times \frac{1}{2})$ sq. in. $= 52.5$ sq. in.

760. The volume $= (12.3 \times 5.72 \times 12 \times \frac{1}{3})$ sq. in. $= 281.42$ sq. in.

761. Let P, P' denote the volumes of the pyramids; then $P : P' = 6.4 \times 144 \times 9 \times \frac{1}{3} : 8.1 \times 144 \times 8 \times \frac{1}{3} = 8 : 9$.

762. The volume $= \{(8 + 4\frac{1}{2} + 6) \times 5 \times \frac{1}{3}\}$ cu. ft. $= 30\frac{5}{6}$ cu. ft.

763. $400 = (20 + 7.2 + \sqrt{20 \times 7.2}) \times h \times \frac{1}{3} = \frac{89.2}{3} h$; $\therefore h = \frac{1200}{89.2} = 30.61$. *Ans.* 30.61 in.

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764. Since each edge is parallel to the intersecting plane, the intersections of the plane with the faces of the prism are parallel to each edge (449), and therefore to each other. Also the intersections of the plane with the bases are parallel (451); hence the section is a parallelogram (131).

765. Let A, A' denote the lateral areas of two right prisms whose common altitude is H, and the perimeters of whose bases are p, p' resp. Since $A = p \times H$, and $A' = p' \times H$, $A : A' = p \times H : p' \times H = p : p'$.

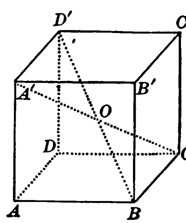
766. Each diagonal of the parallelopiped is equal to the hypotenuse of a right triangle having as arms a diagonal of the base and an edge or altitude of the parallelopiped. But, the base being rectangular, the diagonals of the base are equal; hence the diagonals of the parallelopiped are equal.

767. As shown in Ex. 766, the square of each diagonal of the parallelopiped is equivalent to the sum of the squares of an edge perpendicular to the base and a diagonal of the base. Since the base is rectangular, the square of each of its diagonals is equivalent to the sum of the squares of any two intersecting sides of the base. Hence the square of a diagonal of a parallelopiped is equivalent to the sum of the squares of any three edges that meet in a vertex.

768. Let H denote the altitude of the triangular prism P; B, its base; E, any edge of the base; and D, the distance of E from the opposite vertex. $P = B \times H$ (535), and $B = \frac{1}{2} E \times D$; $\therefore P = \frac{1}{2} E \times H \times D$, $E \times H$ being the area of the lateral face whose edge is E.

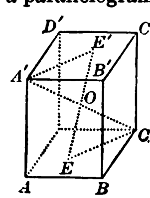
769. Any oblique prism is equivalent to a right prism having a right section as base, and an edge as altitude (519). Hence the volume of the prism is equal to the product of a right section by an edge.

770. Let $ABCD\text{-}C'$ be a parallelopiped, of which $A'C$, BD' are any two diagonals. Since $A'D'$ is parallel to BC, each being parallel to $B'C'$, a plane can be passed through $A'D'$, BC, which will contain both diagonals which will intersect in some point O. Since $A'D'$ and BC are equal as well as parallel, $A'C$, BD' are diagonals of the parallelogram that could be formed by joining $A'B$, CD'. Hence these diagonals bisect each other in O; and as $A'C$, BD' are any two diagonals, all the diagonals bisect each other.

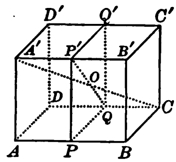


771. Let $A'C$, BD' be two of the intersecting diagonals. Since they intersect, they lie in the same plane; $\therefore$ the edges $A'D'$, BC lie in the same plane. But AD is $\parallel$ to $A'D'$; $\therefore AD$ is $\parallel$ to the plane of $A'D'$ and BC, and as it cannot meet BC, though in the same plane, AD is $\parallel$ to BC. Also, AB is $\parallel$ to CD; $\therefore ABCD$ is a parallelogram; $\therefore ABCD\text{-}C'$ is a parallelopiped (520).

772. Let EE' be any intercept between opposite faces AC, $A'C'$, and passing through O, the mid point of diagonal $A'C$. Join EC, $E'A'$. Since AC', EE' intersect, they are in the same plane; hence EC is $\parallel$ to $E'A'$ (451); $\therefore \angle OEC = \angle OE'A'$; $\therefore \triangle OEC = \triangle OE'A'$ (63); $\therefore OE = OE'$.

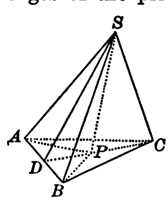


773. Let plane PQ' pass through O, the center of parallelopiped AC'. The figure $PP'Q'Q$ is a parallelogram (451); hence O being equidistant from P' and Q (Ex. 772), the diagonal $P'Q$ of parallelogram PQ', and the diagonal $A'C$ of the parallelopiped, will bisect each other in O; $\therefore \triangle A'OP' = \triangle COQ$ (66); $\therefore A'P' = CQ$; $\therefore B'P' = DQ$. In the same way may be shown that $D'Q' = BP$, and $C'Q' = AP$. Hence if the figure $ADD'A'-Q$ be applied to $BCC'B'-Q'$ so that the equal faces AD' and BC' coincide, the equal perpendiculars at their angular points will coincide, and the figures coincide and be equal.

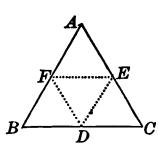


774. (1) If the intersection of the diagonal planes is a lateral edge of the prism, then, since the planes are perpendicular to the base, their intersection is perpendicular to the base (475); hence this lateral edge being perpendicular to the base, all the edges are perpendicular to the base, and the prism is a right prism (506).

(2) If the diagonal planes passing through two edges AA', BB', intersect in an edge PP' without the prism, it is easily shown that $\triangle A'P'B'$ being equal to $\triangle APB$, $A'P'$ is parallel and equal to AP; whence AA' is parallel and equal to PP'; whence AA' and all the edges of the prism are perpendicular to the base.

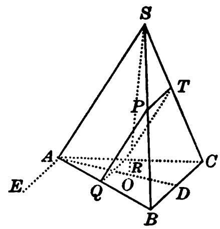


775. Let S–ABC be any pyramid, and SP its altitude. Join P with A, B, C; draw PD perpendicular to AB, and join SD. SD is perpendicular to AB (441). Since SPD is a right angle (Hyp.), $SD > PD$; $\therefore \triangle SAB > \triangle PAB$ (333). Similarly $\triangle SAC > \triangle PAC$, $\triangle SBC > \triangle PBC$; etc. Hence the sum of the faces is greater than the base.



776. Let ABC be one of the four equilateral triangles that form the faces of a regular tetrahedron. If we suppose planes passed through the mid points of the edges, as D, E, F, their intersections with the faces, as EF, will each be equal to one half of the parallel edge, as BC (148). Hence the sections formed by the planes will each be an equilateral triangle equal to the equilateral triangle, as DEF, left on each face. As there are four trihedral angles to be replaced by such triangles, and four faces, the solid formed would be bounded by eight equal equilateral triangles having their vertices at the mid points of the edges of the tetrahedron.

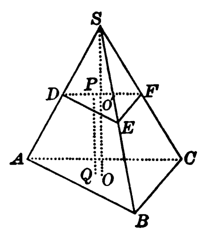
777. Let S–ABC be a triangular pyramid, and $PQRT$ a section formed by a plane passed parallel to SA and also to BC. Since the plane is parallel to SA, PQ and TR are each parallel to SA (449);



and since the plane is parallel to BC, PT and QR are each parallel to BC. Hence PR is a parallelogram (131).

778. In the diagram for Ex. 777 let $S\text{-}ABC$ be a regular tetrahedron, and PR a section formed by a plane $\parallel$ to SA and to BC. Then PR is a parallelogram (Ex. 777). Draw AD to BC at its mid point; then AD is $\perp$ to BC (75), and a plane passed through SA, SD will be $\perp$ to BC, and therefore to ABC (470). Hence if SO be drawn $\perp$ to AD, it will be $\perp$ to ABC. Through A draw $AE \parallel$ to BC and QR, and therefore $\perp$ to AD. Then, since from the foot of the $\perp$ SO, OA is drawn $\perp$ to AE, SA is $\perp$ to AE (441). But PQ is $\parallel$ to SA, and QR to AE; $\therefore$ PQR is a right angle (445); $\therefore$ PR is a rectangle.

779. Let P be the given point within the regular tetrahedron $S\text{-}ABC$, of which SO is the altitude to ABC. Through P pass a plane parallel to ABC, making the triangular section DEF, and cutting SO in O'. Since the sides of $\triangle DEF$ are respectively parallel to those of $\triangle ABC$, DEF is equilateral, as are also all the faces SDE, etc. Hence there is cut off a second regular tetrahedron $S\text{-}DEF$, whose altitude $SO' = SO - PQ$, PQ being the perpendicular from P to ABC. If a plane be passed through P parallel to a second face of $S\text{-}ABC$, there will be cut off the second one a third regular tetrahedron. Denoting by SO'' its altitude, and by PQ' the perpendicular from P to the second face, we have $PQ' = SO' - SO''$. If a plane again be passed through P parallel to a third face of $S-ABC$, a fourth regular tetrahedron will be cut off from the third one, and $PQ'' = SO'' - SO'''$. But the altitude from P to a fourth face is equal to the altitude SO''' of the tetrahedron, since P is now a vertex; *i.e.*, $PQ''' = SO'''$. Hence



$$PQ + PQ' + PQ'' + PQ'''$$
$$= SO - SO' + SO' - SO'' + SO'' - SO''' + SO''' = SO.$$

EXERCISES, pp. 294–329.

780. If, besides the element that passes through the given point, a second straight line could be drawn on the cylindrical surface, that surface would coincide with the plane determined by those lines.

781. If the line joining the given points did not coincide with an

element, there would be two straight lines drawn through each of the points, which cannot be (Ex. 780).

782. For if the plane intersected the cylindrical surface, it would either intersect all the elements or coincide with two of them.

783. Since the cylinder is supposed to roll without slipping, the elements will remain parallel to their original position as they coincide in succession with the plane. Hence the figure described will be a rectangle whose altitude $= a$, whose base $= 2\pi \cdot r$, the circumference of the base, and whose area $= 2\pi \cdot r \cdot a$.

784. Since the cone is supposed to roll without slipping, the vertex will retain its original position while each point of the circumference of the base coincides in succession with the plane. The elements being of equal length, the line in which their basal extremities meet the plane will be a circular arc whose length $= 2\pi \cdot r$, the circumference of the base. The radius of this arc being $\sqrt{a^2 + r^2} =$ the slant height of the cone, the area of the sector formed will be $2\pi \cdot r \times \frac{1}{2}\sqrt{a^2 + r^2}$ $= \pi \cdot r \cdot \sqrt{a^2 + r^2}$.

785. In order that the rectangle whose numerical measure is $2\pi \cdot r \cdot a$ (Ex. 783) shall be an exact square, we must have $2\pi \cdot r = a$; $\therefore\ r = a \div 2\pi$.

786. Since, slipping excluded, the vertex retains its position, while the axis has always the same inclination to the plane, viz., its inclination to any element, the basal extremity of the axis will describe a circumference, and the axis will generate a conical surface (576).

787. Since the vertex of the cone retains its position, while its basal extremity, always at a certain distance from the surface of the fixed cone, describes a circumference, the axis will generate a conical surface, each of whose elements makes a constant angle with an element of that cone.

788. The radius of the base B being r, that of S, the circle generated, is equal to the slant height of the cone $= \sqrt{a^2+r^2}$; $\therefore\ B : S = r^2 : a^2 + r^2$.

789. The arc of the sector is equal to the circumference of the base $= 2\pi \cdot r$, and the circumference of the circle generated $= 2\pi\sqrt{a^2 + r^2}$. Now the sector is to the circle as its arc is to the circumference (403); hence sector : circumference $= r : \sqrt{a^2 + r^2}$.

790. The circumference of the base of the cone $= \frac{300}{360} C$, the circumference of the given circle, $= \frac{5}{6} C$; $\therefore\ r = \frac{5}{6} R$ (393). The slant height of the cone $= R$; $\therefore\ a = \sqrt{R^2 - r^2} = R \times \sqrt{\frac{11}{36}}$.

791. Since the circumference c of the base of the cone $= \frac{m}{n} C$, $r = \frac{m}{n} R$ (393), and $a = \sqrt{R^2 - \left(\frac{m}{n} R\right)^2} = R \times \sqrt{\frac{n^2 - m^2}{n^2}}$.

792. Let ABC be a spherical triangle having B and C each at a

quadrant's distance from A. A is the pole of arc BC (600); $\therefore \angle A$ is measured by BC (616).

793. In the diagram for Prop. XIV. let $A'B'C'$ be the polar triangle of $\triangle ABC$. Then (632) $B'C'=180°-83°=97°$; $A'C'=180°-50°=130°$; $A'B'=180°-97°=83°$.

794. In the same diagram suppose $\angle B=\angle C$; then $A'C'=180°-\angle B$, and $A'B'=180°-\angle C=180°-\angle B$; $\therefore A'B'=A'C'$.

795. The acute angles being equal, the triangles are mutually equiangular; hence they are mutually equilateral (647), and, being birectangular, are isosceles, and therefore equal (639).

796. Let the birectangular $\triangle$s ABC, $A'B'C'$, on spheres whose radii are r, r' respectively, have acute $\angle A$ equal to acute $\angle A'$. The triangles are mutually equiangular. Now BC, $B'C'$ have each the same number of degrees as $\angle A$ and $\angle A'$, say $a°$ (Ex. 795). But in BC $a° = a$ times $\frac{2\pi r}{360}$, and in $B'C'$ $a° = a$ times $\frac{2\pi r'}{360}$; $\therefore BC:B'C'=r:r'$, $AB:A'B'=r:r'$, $AC:A'C'=r:r'$; $\therefore AB:AC:BC=A'B':A'C':B'C'$, and the triangles are similar (284).

797. For each is one eighth of each of the equal spherical surfaces.

798. The triangles are mutually equiangular (Hyp.), and the sides are proportional, each being one fourth of a great circle of its own sphere; hence the triangles are similar (284).

799. In the diagram for Prop. XIV, let $A'B'C'$ be the polar triangle of $\triangle ABC$, having B and C right angles. Since $\angle B=\angle C=90°$, $A'B=A'C'=90°$ (632); $\therefore A$ is the pole of $B'C'$; $\therefore \angle B'$ and $\angle C'$ are right angles (618).

800. In the same diagram, if $\angle B=\angle C=90°$, then $\angle B'=\angle C'=90°$ (Ex. 799); $AB=180°-\angle C'=90°$, $AC=180°-\angle B'=90°$.

801. This follows from Ex. 800, or from Art. 648.

802. Each side being a quadrant (Ex. 801), each vertex is the pole of the opposite side.

803. Since the sum of the angles of a spherical triangle is greater than two right angles (633), if one angle be a right angle, the sum of the other two must be greater than a right angle.

804. Since the angles of the lunes are equal, the birectangular triangles into which each may be divided are equiangular, and hence similar.

805. Denoting by L and S respectively the lune, and the spherical surface of which it is a part, then (657) $L:S=36:360=1:10$.

806. Let x denote the number of degrees in the angle of the lune; then (657) $x:360=\frac{1}{100}:1$; $\therefore x=3.6$.

807. Since the lune $=\frac{5}{12}T=\frac{5}{96}S$, then, denoting by x the number of degrees, $x:360=\frac{5}{96}:1$; $\therefore x=\frac{75}{4}$. *Ans.* angle $=18°\,45'$.

808. $U : V = 25 : 360$; $\therefore$ $U = \frac{5}{72} V$ (660).

809. Let x denote the number of degrees in the angle of the lune that is the base of the wedge; then (660) $x : 360 = \frac{1}{36} : 1$; $\therefore$ $x = 10$.

810. Denoting by E the spherical excess of polygon P, $E = (140 + 90 + 93 + 120 + 117 - 180) = 380$; $\therefore$ (665) $P : S = 380 : 720$; $\therefore P = \frac{19}{36} S$.

QUESTIONS, p. 330.

811. The required locus is the circular cylindrical surface having the given line as axis, and a radius equal to the given distance.

812. The cylinder may be regarded as the limiting form to which a prism tends as the number of its faces becomes indefinitely great.

813. The required locus is a conical surface, having the given line as axis, and each element making with that axis an angle equal to the given angle.

814. The cone may be regarded as the limiting form to which a pyramid tends, as the number of its faces becomes indefinitely great.

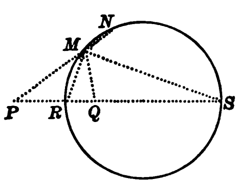
815. The required locus is the surface of the sphere having the given point as center, and a radius equal to the given distance.

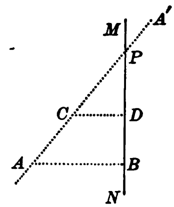
816. The spherical angles being equal, the dihedral angles formed by the planes of their sides are equal; hence the trihedral angles having their vertices at the center are equal or symmetrical; hence the face angles of those trihedrals are respectively equal, and, on equal spheres, are subtended by arcs that are respectively equal.

817. They will form a right angle, since the section determined by the plane of those lines and the diameter will be a semicircle.

818. This plane is the locus of all straight lines that are tangent to the sphere at that point.

819. At any point P in one of the parallels pass a plane $KL \perp$ to them, and meeting the second one in Q. Join PQ, divide PQ in R in the given ratio, and produce to S so that $PS : QS = PR : RQ$ (309). The circumference described in plane KL with RS as diameter, is the locus of all points in KL whose distances from P and Q are in the given ratio (Ex. 456). Hence it is apparent that the required locus is the cylindrical surface having as directrix the circumference whose diameter is RS, and whose elements are perpendicular to the plane of that circle, *i.e.*, are parallel to the given lines. For any right section of this cylindrical surface will be a circle, all the points in whose circumference have their distances from the points where the plane cuts the given lines, in the given ratio.





820. Let P be the given point in the given line MN, and A a point in the required locus. Through AP draw the indefinite line AA', and draw AB perpendicular to MN. At any other point C in AA', draw CD perpendicular to MN; then $CP : CD = AP : AB$; *i.e.*, any point in AA' is a point in the required locus. If now AA' be revolved about MN so that A always remains at the distance AB from MN, the circular conical surface thus generated is the required locus, since each point in every element of that surface has its distances from P and MN in the given ratio.

821. The locus consists of two planes parallel to the given plane at the given distance, one on each side.

822. Let d be the given distance, and r the radius of the sphere. If $d < r$, the required locus will consist of two concentric spherical surfaces, whose radii are $r + d$, $r - d$ respectively. If $d > r$, the locus will be the concentric spherical surface whose radius is $r + d$.

823. Let d be the given distance, and r the radius of the directrix. If $d < r$, the required locus will consist of two circular cylindrical surfaces, having the same axis as the given surface, and having $r + d$, $r - d$ respectively as radii of their directrices. If $d > r$, the locus will be the cylindrical surface the radius of whose directrix is $r + d$.

824. The required locus will consist of two circular conical surfaces, having the same axis as the given surface, and having their elements parallel to those of the given surface, at the given distance.

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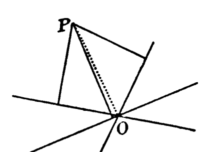
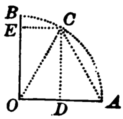
825. The locus of all points such that lines drawn from each to the extremities of a given line form a right angle, is the circumference described on that line as diameter; and the locus of all circumferences having that line as diameter is the surface of the sphere generated by revolving that surface. This sphere is the required locus.



826. Let O be the point through which pass the given lines, and P the given point from which perpendiculars are drawn to them. Join PO. Then PO is the common hypotenuse of all the right triangles in space whose vertices are the feet of the perpendiculars to the given lines. Hence these feet lie on the surface of a sphere whose diameter is PO (Ex. 825).

827. Let O be the point through which pass the given lines, all in the same plane MN, and P a point without MN. It may be shown

(Ex. 826) that the feet of the perpendiculars lie on the surface of a sphere, but being all in one plane, lie in a circumference.

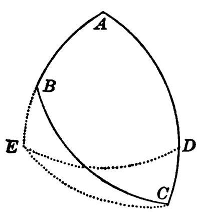


828. Let OA, OB be radii perpendicular to each other, and subtended by the quadrant ACB, of which BC is one third. Draw CD, $CE \perp$ to OA, OB respectively, and join CA, CO. Since $AC = \frac{2}{3}$ quadrant, $AC = OA = OC$; $\therefore OD = \frac{1}{2} OA$; $\therefore EC = \frac{1}{2} OA$. Hence if the quadrant ACB be revolved about OB as axis, OA will describe a great circle, and EC, a circle of one half the radius.

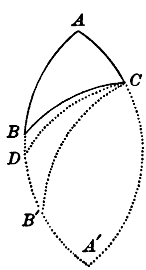
829. A trirectangular triangle is equivalent to a lune whose angle is 45°. Let A be the angle of the lune; then $L : T = A : 45°$.

830. Let E, E' denote the spherical excesses of polygons P, P' respectively, on the sphere whose surface is S. Since

$$P \approx \frac{S}{720} \times E, \text{ and } P' \approx \frac{S}{720} \times E' \text{ (664)}, \quad P : P' = E : E'.$$



831. In spherical $\triangle ABC$, having A a right angle, let $AC > 90°$, $AB < 90°$, and $BC < AC$. Since $AC > BC$, $\angle B > \angle A$ (651); *i.e.*, $\angle B$ is obtuse. From A as pole, describe an arc of a great circle meeting AC in D, and AB produced in E. Join CE by an arc of a great circle. Since ADE is trirectangular, E is the pole of arc AC; $\therefore CE$ is a quadrant. But $\angle CEB >$ rt. $\angle DEA$, and $\angle CBE <$ a right angle, since $\angle B$ is obtuse; $\therefore CB > CE$; *i.e.* CB is greater than a quadrant.



832. (1) Let ABC be a spherical triangle having A a right angle, and AB less than a quadrant. Produce AB, AC to meet in A'. Then ABA', ACA' are semicircumferences, which are $\perp$ to each other, A being a right angle. Take D the mid point of ABA'. D is the pole of arc ACA' (600). Join DC. DC is a quadrant, and ACD a right angle; $\therefore \angle ACB <$ a right angle.

(2) If in $\triangle AB'C$, $AB' >$ than a quadrant, we prove in the same way that $\angle ACB' >$ a right angle.

833. In the rt. spher. $\triangle$s ABC, $A'B'C'$, let hypot. $BC =$ hypot. $B'C'$, and $\angle B = \angle B'$. According as $\angle$s B and B' are, or are not, similarly situated in the two triangles, we can place $\triangle ABC$ upon $\triangle A'B'C'$, or upon the symmetrical $\triangle$ of $A'B'C'$, and prove as in Prop. XII., Book I.

834. In the rt. spher. $\triangle$s ABC, $A'B'C'$, let hypot. $BC =$ hypot. $B'C'$, and arm $AB =$ arm $A'B'$. According as AB and $A'B'$ are, or are not, similarly situated in the two triangles, we can apply $\triangle ABC$ to $\triangle A'B'C'$, or to the symmetrical triangle of $A'B'C'$, so that BC will

coincide with $B'C'$, but A, A' lie on opposite sides of $B'C'$, and prove as in Prop. XI., Book I.

835. Let AD be the arc bisecting spher. $\angle BAC$. Join A, B, C, D, with O, the center of the sphere. Then OAD is the plane bisecting the dihedral angle whose edge is OA. But OAD is the locus of all points equidistant from the planes AOB, AOC. Hence if from any point P in AD arcs be drawn $\perp$ to AB, AC, resp., the planes of these arcs being $\perp$ to the planes AB, AC, the chords of these arcs will be $\perp$ to these planes, and therefore be equal; hence the arcs are also equal, and these arcs measure the distances from P to AB and AC.

EXERCISES, pp. 335–340.

836. $AB=10$, $AC=5$, $AD=3$, $BE=1$, and by similar triangles $CF : BE = 5 : 10$; $\therefore\ CF = BE \times .5 = .5$; $DG : BE = 3 : 10$; $\therefore\ DG = BE \times .3 = .3$. Hence $M : N = 30 + 5 + .5 : 10 + 1 + .3 = 35.5 : 11.3 = 355 : 113$.

837.

$$\begin{aligned} \text{Since } 355 \div 113 &= 3.141592920+ \\ \text{and } \pi &= \underline{3.141592654-} \\ \text{dif.} &= \ \ .000000266 \end{aligned}$$

Hence the line will be too great by .000000266 yd.; *i.e.*, by less than .0000096 in.

838. $S = \frac{1}{2} \times 12\,(\pi \times 10 + \pi \times 8) = 6 \times \pi \times 18 = 339.29$ sq. in.

839. Section $= \pi \times \frac{1}{4}\left(\dfrac{10+8}{2}\right)^2 = \pi \times 20.25 = 63.62$ sq. in.

840. $V = \frac{1}{3}\,\pi \times 10^2 \times 20 = 2094.4$ cu. in.

841. Since similar cones are as the cubes of any of their homologous lines, $V : V' = H^3 : H'^3 = 1 : 2$; $\therefore\ H' = H \times \sqrt[3]{2}$.

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842. Circumference of base $= (\pi \times 12)$ in.; $\therefore$ lat. area $= (12\,\pi \times 18)$ sq. in. $= 678.59$ sq. in.

$$\text{Area of base} = (\pi \times 36) \text{ sq. in.} = 113.10 \text{ sq. in.};$$

$$\therefore \text{ total area} = (678.56 + 113.10 \times 2) \text{ sq. in.} = 904.79 \text{ sq. in.}$$

843. Area of base $= 113.10$ sq. in.; $\therefore$ vol. $= (113.10 \times 18)$ cu. in. $= 2035.80$ cu. in.

844. If R denote the radius, then by the conditions,

$2\,R \times \pi \times 14 + R^2 \times \pi \times 2 = 700$; $\therefore\ R^2 + 14\,R = 700 \div 2\,\pi = 111.4082$. Hence, solving as a quadratic, $R = 5.66$ in.; $\therefore\ D = 11.32$ in.

845. Let H denote the altitude; then by the conditions,

$$\pi \times 5 \times H = 144;\ \therefore\ H = 9.17 \text{ in.}$$

846. Let R denote the radius; then by the conditions,

$$2\pi \times R \times 10 + R^2 \times \pi \times 2 = 500;$$

$$\therefore\ R^2 + 10R = 500 \div 2\pi = 79.577;\ \therefore\ R = 5.23.$$

847. Let R denote the radius; then by the conditions,

$2R \times \pi \times 6R + R^2 \times \pi \times 2 = 1200;\ \therefore 7R^2 = 1200 \div 2\pi = 190.9855;$

$$\therefore R = 5.223;\ \therefore D = 10.45 \text{ in.},\ H = 31.35 \text{ in.}$$

848. Let T and T' denote the total areas, V and V' the volumes; then $T : T' = 7^2 : 5^2 = 49 : 25$; $V : V' = 7^3 : 5^3 = 343 : 125$.

849. Since the formula for the total area is

$$T = 2R \times \pi \times H + R^2 \times \pi \times 2, \text{ if } H = R,\ T = 4\pi R^2.$$

850. Since $V = R^2 \times \pi \times H$, if $H = R$, then $V = \pi R^3$.

851. Let V' denote the volume of the cube whose edge $= R$;

$$\text{then } V : V' = \pi R^3 : R^3 = \pi : 1.$$

852. The circumference of the base $= 12\pi$, area $= 36\pi$, slant height $= \sqrt{15^2 + 6^2} = 16.1555$; $\therefore$ lateral area $= \frac{1}{2}(12\pi \times 16.1555)$ sq. in. $= 304.52$ sq. in. Total surface $= \frac{1}{2}(12\pi \times 16.1555 + 36\pi \times 2)$ sq. in. $= 417.62$ sq. in.

853. Vol. $= \frac{1}{3}\pi \cdot 6^2 \cdot 15 = 565.49$ cu. in.

854. If R denote the radius, the slant height $= \sqrt{R^2 + 100}$. Then the total area $= 2R \times \pi \times \frac{1}{2}\sqrt{R^2 + 100} + R^2 \times \pi = 400$;

$\therefore \sqrt{R^4 + 100R^2} + R^2 = 400 \div \pi = 127.3236$; $\therefore R = 6.76$, $D = 13.52$ in.

855. If H denote the altitude, the slant height $= \sqrt{H^2 + 25}$.

$$\text{The lateral surface} = 10 \times \pi \times \tfrac{1}{2}\sqrt{H^2 + 25} = 144;$$

$$\therefore \sqrt{H^2 + 25} = 144 \div 5\pi = 9.167;\ H = 7.68 \text{ in.}$$

856. If R denote the radius, the slant height $= \sqrt{R^2 + 100}$.

$$\text{But } T = 2R \times \pi \times \tfrac{1}{2}\sqrt{R^2 + 100} + R^2 \times \pi = 100;$$

$$\sqrt{R^4 + 100R^2} + R^2 = 31.83;\ \therefore R = 2.49.$$

857. If R denote the radius, $4R$ will denote the altitude, and $\sqrt{R^2 + 16R^2} = R\sqrt{17}$, the slant height. The lateral area $= 2R \times \pi \times \frac{1}{2}R\sqrt{17} = 500$;

$$\therefore R^2 = 500 \div \pi\sqrt{17} = 38.6;\ \therefore R = 6.21,\ H = 24.84.$$

858. Let T, T' denote the total areas, and V, V' the volumes; then

$$T : T' = 11^2 : 8^2 = 121 : 64, \text{ and } V : V' = 11^3 : 8^3 = 1331 : 512.$$

859. If H the altitude $= R$ the radius, the slant height $= \sqrt{R^2 + R^2}$ $= R\sqrt{2}$; $\therefore T = 2R \times \pi \times \frac{1}{2}R\sqrt{2} + R^2 \times \pi = \pi \cdot R^2(\sqrt{2} + 1)$.

860. $V = R^2 \times \pi \times \frac{1}{3}R = \frac{1}{3}\pi \cdot R^3$.

861. The base of the tetrahedron is an equilateral triangle whose side is equal to $2R \div \sqrt{3}$. If V' denote the volume of the tetrahedron,

$$V' = R^2 \times \frac{\sqrt{3}}{4} \times R \times \tfrac{1}{3} = \frac{\sqrt{3}}{12}R^3;$$

$$\therefore\ V : V' = \tfrac{1}{3}\pi \cdot R^3 : \frac{\sqrt{3}}{12}R^3 = \pi : \frac{\sqrt{3}}{4}.$$

862. The lateral area $= 2\,R \times \pi \times \frac{1}{2}\,R\sqrt{2} = 100$; $\therefore R^2 = 100 \div \pi\sqrt{2}$; $\therefore R = 4.75$ in. $= H$.

863. $V = \frac{1}{3}\pi \cdot R^3 = 1000$; $\therefore R^3 = 954.93$; $\therefore R = 9.85$ in. $= H$.

864. In the diagram for the corollary of Prop. IV., let OO' be the altitude of the frustum. Through OO' pass a plane intersecting the bases in OA, $O'A'$ respectively, and from A' suppose $A'B$ drawn $\perp$ to OA. Then $\overline{AA'}^2 = \overline{A'B}^2 + \overline{AB}^2 = H^2 + (R - R')^2$;

$$\therefore AA' = \sqrt{20^2 + (7-3)^2} = \sqrt{416} = 20.4 \text{ in.}$$

Hence the lateral area $= \frac{1}{2} \times 20.4\,(14\,\pi + 6\,\pi) = 640.9$ sq. in. $T = 640.89 + (7^2 + 3^2)\,\pi = 823.10$ sq. in.

865. $V = \frac{1}{3} \times 20 \times (49\,\pi + 9\,\pi + 21\,\pi) = 1654.58$ cu. in.

866. $\frac{1}{3}H\,(25\,\pi + 64\,\pi + 40\,\pi) = 575$; $\therefore H = 1725 \div 129\,\pi = 4.25$ in.

867. Let X denote the altitude of the cone cut off by the plane; then $X^3 : 16^3 = 1 : 2$ (694); $\therefore X = 16 \div \sqrt[3]{2} = 12.7$ in.; $\therefore 16 - X = 3.3$ in.

868. Let S, S' denote the lateral surfaces of the cylinder and cone respectively, V and V' the volumes, H being the altitude, and R the radius of the base, in each. Then since $L = \sqrt{R^2 + 9\,R^2} = R\sqrt{10} = \frac{1}{3}H\sqrt{10}$,

$$S : S' = 2\,R \times \pi \times H : \tfrac{1}{2}\,(2\,R \times \pi \times \tfrac{1}{3}\,H\sqrt{10}) = 6 : \sqrt{10}.$$

$$V : V' = R^2 \times \pi \times H : \tfrac{1}{3}\,R^2 \times \pi \times H = 3 : 1.$$

869. The volume of the cylinder $= (25\,\pi \times 7) = 549.78$ cu. ft.;

$\therefore$ the edge of the equivalent cube $= \sqrt[3]{549.78} = 8.19$ ft.

870. The slant height of the cone $= \sqrt{(5)^2 + 15^2} = 15.81$ in.;

$\therefore$ the lateral surface $= \pi \times 100 \times 7.9 = 2481.86$ sq. in.

Let D denote the diameter of the required cylinder; then

$$D \times \pi \times 15 = 2481.86; \therefore D = 52.66 \text{ in.}$$

871. Let R denote the radius of the hollow cylinder, and R' that of the empty cylindrical space, H being the common altitude; then

$$R^2 \times \pi \times H = 2\,(R^2 - R'^2) \times \pi \times H; \therefore R^2 = 2\,R'^2; \therefore R : R' = \sqrt{2} : 1.$$

872. Let R be radius of tank to contain $231 \times 2000 = 462000$ cu. in. Since $R^2 \times \pi \times 144 = 462000$, $R = 31.96$ in.; $\therefore D = 5.33$ ft.

873. The outer diameter of the tank will be $63.92 + 2 = 65.92$ in., and its height 145 in. Denoting by V and V' the volumes of this tank and the inner cylinder respectively, we find $V - V' = (32.96)^2 \times \pi \times 145 - 462000 = 32872.5$ cu. in.

EXERCISES, pp. 346-349.

874. Since $R = 6$, the area of a great circle $= \pi \times 6^2 = 113.10$ sq. in.; hence the surface of the sphere $= (113.10 \times 4)$ sq. in. $= 452.40$ sq. in.

875. $D^2 \times \pi = 400$; $\therefore D = \sqrt{400 \div \pi} = 11.28$ in.

876. $S : S' = 10^2 : 12^2 = 25 : 36$.

877. $Z = H \times 2\pi \times R = 3 \times 2\pi \times 6 = 113.10$ sq. in.

878. $Z = H \times 2\pi \times R = \frac{1}{n} \times 4\pi \times R^2;\ \therefore H = R \times \frac{2}{n} = \frac{D}{n}.$

879. $V = \frac{1}{6}\pi \cdot D^3 = \frac{1}{6}\pi \times 1 = .5236$ cu. ft.

880. $\frac{1}{6}\pi \cdot D^3 = 1728;\ \therefore D = \sqrt[3]{1728 \times 6 \div \pi} = 14.89$ in.

881. By Ex. 880, $D = 14.89$; $\therefore R = 7.445$;

$\therefore S = (7.445)^2 \times 4\pi = 696.53$ sq. in.

882. $S : S' = 10^2 : 12^2 = 25 : 36.\quad V : V' = 10^3 : 12^3 = 125 : 216.$

883. $V : V' = R^3 : R'^3 = 2 : 1;\ \therefore R : R' = \sqrt[3]{2} : 1.$

884. $V = \frac{2}{3}\pi \cdot R^2 \cdot H = \frac{2}{3}\pi \times 25 \times 2 = 104.72$ cu. in.

885. Since the base of the given pyramid is $\frac{1}{8}$ of the surface of the sphere (648, 714), the pyramid is equal to $\frac{1}{8}$ of the sphere.

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886. $S = (3.5)^2 \times 4\pi = 153.94$ sq. in.

887. $R^2 \times 4\pi = 100$; $\therefore R = 2.82$ in.

888. Denoting the surfaces by S, S', and the volumes by V, V',

$S : S' = 9^2 : 5^2 = 81 : 25;\ V : V' = 9^3 : 5^3 = 729 : 125.$

889. Denoting the radii by R, R', and the volumes by V, V', $R^2 : R'^2 = 125 : 27;\ \therefore R : R' = 1 : .465.\quad V : V' = 1 : (.465)^3 = 1 : .1005.$

890. The altitude of the zone $= (13 \mp 9)$ in., according as the planes are or are not on the same side of the center; $\therefore Z = (4 \times 18 \times 2\pi)$ sq. in. $= 452.39$ sq. in., or $= (22 \times 18 \times 2\pi)$ sq. in. $= 2488.15$ sq. in.

891. Vol. of spher. sector $= (452.39 \times 18 \times \frac{1}{3})$ cu. in. $= 2714.34$ cu. in.

892. $Z = 2 \times 12 \times \pi \times H = 100$; $\therefore H = 100 \div 75.4 = 1.33$ in.

893. Since the spherical surface $= (12)^2 \times 4\pi = 1809.56$ sq. in., and the zone $= 2 \times 12 \times \pi \times H = 1809.56 \div 4$, $H = 6$ in.;

or (705) $H : 12 =$ zone : hemisphere $= 1 : 2$; $\therefore H = 6$ in.

894. (1) $V = \frac{1}{6}\pi \times 1^3 = .5236$ cu. ft. (2) $V = \frac{1}{6}\pi \times (1.5)^3 = 1.767$ cu. ft.

895. Since $R^2 \times 4\pi = 64$, $R = 2.26$; $\therefore V = 48.21$ cu. in.

896. Since $\frac{4}{3}\pi \cdot R^3 = 1728 \times 5 = 8640$, $R = 12.73$; $\therefore D = 25.46$ in.; also $S = (12.73)^2 \times 4\pi = 2036.42$ sq. in.

897. Let V, V' denote the volumes; then

$V - V' = 12^3 \times \frac{4}{3}\pi - 7^3 \times \frac{4}{3}\pi = \frac{4}{3}\pi(1728 - 343) = 5801.49$ cu. in.

898. $V : V' = R^3 : R'^3 = 27 : 8$; $\therefore R : R' = 3 : 2$, and $D : D' = 3 : 2$. Also $S : S' = R^2 : R'^2 = 9 : 4$.

899. $V : V' = 4^3 : 5^3 = 64 : 125.\quad S : S' = 4^2 : 5^2 = 16 : 25.$

900. $Z = 2\pi \times R \times H = \pi \cdot R^2$; $\therefore H = \frac{1}{2}R = 3$ in.

901. $Z = 2\pi \times 12 \times H = 50$; $\therefore H = .66$ in.; also sector $= \frac{1}{3} \times 50 \times 12 = 200$ cu. in.

902. Sector $= \frac{1}{3}Z \times 7 = 25$; $\therefore Z = 10.71$ sq. in.

903. $S=(2.5)^2\times 4\pi=78.54$ sq. in. $V=\frac{1}{3}(78.54\times 2.5)=65.45$ cu. in.

904. $\frac{4}{3}\pi\cdot R^3=1728$; $\therefore$ $R=7.444$, $D=14.89=$ diameter and altitude of cylinder; $\therefore$ lateral surface of cylinder $=\pi(14.89)^2=696.53$ sq. in.

905. If $\frac{4}{3}\pi\cdot R^3=4\pi\cdot R^2$, then $R=3$, $D=6$.

906. Since the edge of the cube $=\sqrt[3]{1331}=11$, $R=5.5$;

$$\therefore\ V=(5.5)^3\times\frac{4}{3}\pi=696.91 \text{ cu. in.}$$

907. $4\pi\cdot R^2=\pi\cdot D^2=150$; $\therefore$ $D^2=47.746$ sq. in. $=$ a face of the circumscribed cube, since D is equal to an edge; $\therefore$ surface of cube $=$ (47.746×6) sq. in. $=286.48$ sq. in.

908. Let V, V' denote the volumes of the whole shell and of the inner hollow part respectively, the radius of the latter being R. $V-V'=\frac{4}{3}\pi(6^3-R^3)=696.9$. Hence $R^3=49.627$, $R=3.675$ in.; hence the thickness of the wall of the shell $=(6-3.675)$ in. $=2.32$ in.

909. Let V, V' denote the volumes of the spheres, and n' the weight of the second sphere. $V:V'=3^3:4^3=27:64=n:n'$; $\therefore$ $n'=\frac{64}{27}n$.

910. In the diagram for Art. 717, let $OA=10$, $AD=8$, $DC=2$; then $OD=\sqrt{100-64}=6$; $\therefore$ $OC=6+2=8$; $\therefore$ $BC=\sqrt{100-64}=6$; $\therefore$ segment$=\frac{1}{2}\times 2(64\pi+36\pi)+\frac{1}{3}\pi\times 8=314.16+8.38=322.54$ cu. in.

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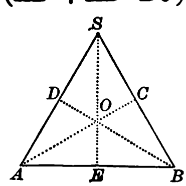
911. Let H denote the altitude, and D the diameter, of the cylinder. The lateral surface $=\pi\cdot D\cdot H=\pi\cdot(\sqrt{D\cdot H})^2=$ the area of a circle whose radius is $\sqrt{D\cdot H}$.

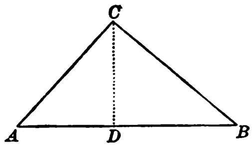
912. Let AB and BC be adjacent sides of the rectangle AC. Then, according as AC is revolved about one side or the other, the generated cylinder will have AB for altitude and BC for radius, or *vice versa*. But $\pi\times 2\,AB\times BC=\pi\times 2\,BC\times AB$.

913. Let T, T' denote the total areas, and V, V' the volumes, generated with AB, BC respectively as axes, and suppose $AB:BC=m:n$. Then $T:T'=2\pi(\overline{BC}^2+BC\cdot AB):2\pi(\overline{AB}^2+AB\cdot BC)$ $=BC:AB=n:m$. $V:V'=\frac{1}{4}\pi\cdot\overline{BC}^2\cdot AB:$ $\frac{1}{4}\pi\overline{AB}^2\cdot BC=BC:AB=n:m$.

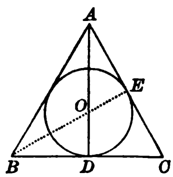


914. Let SAB be a section of the cone through the axis. Since $SA=SB=AB$ (Hyp.), the triangle is equilateral. Draw the altitudes AC, BD, SE, to SB, SA, AB, respectively. Since these altitudes are also medians, they are each divided in O so that $OE=\frac{1}{3}SE$. But $SE=AB\times\frac{\sqrt{3}}{2}$; $\therefore$ OE, the radius of the inscribed sphere, $=AB\times\frac{1}{2\sqrt{3}}$. Let T denote the total area of the cone, and

S that of the inscribed sphere; then $T = AB \times \pi \times \frac{1}{2} SA + \pi \cdot \overline{AE}^2 = \pi \cdot \frac{3}{4}\overline{AB}^2$, and $S = \frac{4}{3}\pi \cdot \overline{AE}^2 = \pi \times \frac{1}{3}\overline{AB}^2$; $\therefore\ T : S = \frac{3}{4}\pi \cdot \overline{AB}^2 : \frac{1}{3}\pi \cdot \overline{AB}^2 = 9 : 4$.



. **915.** Let CAB be a right triangle, having $BC = a$, $AC = b$; then $AB = \sqrt{a^2 + b^2}$, and CD, the altitude to the hypotenuse $= ab \div \sqrt{a^2 + b^2}$. By revolving the triangle about AB as axis, there are generated two cones, having CD as radius of their common base, and slant heights CA, CB respectively. Let S, S' denote the lateral areas of these cones; then $S = \pi \cdot CD \cdot CA$, and $S' = \pi \cdot CD \cdot CB$; $\therefore\ S + S' = \pi \cdot CD(CA + CB) = \pi \dfrac{ab}{\sqrt{a^2 + b^2}}(a + b)$.

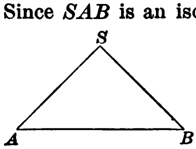


916. Let ABC be the given equilateral triangle, and AD an altitude. Draw a second altitude, BE, intersecting AD in O. O is the center of the inscribed circle, and $OD = BD \times \dfrac{\sqrt{3}}{2}$. Let S, S' denote the surfaces generated. $S = 2\,BD \times \pi \times \frac{1}{2} AB = 2\pi \cdot \overline{BD}^2$. $S' = 4\pi \cdot \overline{OD}^2 = 4\pi \times \overline{BD}^2 \times \frac{3}{4} = 3\pi \cdot \overline{BD}^2$; $\therefore\ S : S' = 2\pi \cdot \overline{BD}^2 : 3\pi \cdot \overline{BD}^2 = 2 : 3$.

917. In the diagram for Ex. 916, OD is the radius of the inscribed circle, OB of the circumscribed circle, $AD = BD \times \sqrt{3}$; $\therefore\ OD = BD \times \dfrac{1}{\sqrt{3}}$, and $OB = BD \times \dfrac{2}{\sqrt{3}}$. Let V, V', V'', denote the volumes generated by the triangle, the inscribed circle, and the circumscribed circle, respectively. Then $V : V' : V'' = \pi \cdot \overline{BD}^2 \times \frac{1}{3} AD : \frac{4}{3}\pi \cdot \overline{OD}^3 : \frac{4}{3}\pi \cdot \overline{OB}^3$

$$= \pi \cdot \overline{BD}^3 \times \frac{1}{\sqrt{3}} : \pi \cdot \overline{BD}^3 \times \frac{4}{9\sqrt{3}} : \pi \cdot \overline{BD}^3 \times \frac{3^2}{9\sqrt{3}} = 9 : 4 : 32.$$

918. Let SAB be a section through the axis of the required cone. Since SAB is an isos. rt. $\triangle$, $AB = SA \times \sqrt{2}$; $\therefore$ the circumf. c of the base of the cone $= \pi \cdot AB = \pi \cdot SA \times \sqrt{2}$. But SA being a radius of the given circle, the circumf. C of this circle $= \pi \cdot SA \times 2$; $\therefore\ C : c = \pi \cdot SA \times 2 : \pi \cdot SA \times \sqrt{2} = \sqrt{2} : 1$. Let n denote the number of degrees in the arc of the sector left to form the cone; then $360 : n = C : c = \sqrt{2} : 1$; $\therefore\ n = 360 \div \sqrt{2} = 254.6$. Hence the number of degrees in the sector cut out is about $105^\circ\ 42'$.



919. If Z, Z' denote the zones, then, each being a zone of one base, $Z : Z' = m : n$ (705).

920. Let V, S, H, R, denote the volume, lateral surface, altitude, and radius, respectively. Then $V = \pi \cdot R^2 \cdot H = \frac{1}{2}(2\pi \cdot R \cdot H \cdot R) = \frac{1}{2}(S \times R)$.

921. Let V, T, H, R, denote the volume, total area, altitude, and radius, respectively.

Then $$V = \pi \cdot R^2 \cdot H = \pi \cdot R^2 \times 2R = \tfrac{1}{3}(6\pi \cdot R^2 \cdot R)$$
$$= \tfrac{1}{3}(2\pi \cdot R \times 2R + 2\pi \cdot R^2)R = \tfrac{1}{3}T \cdot R.$$

922. Let R be the radius of the sphere, V its volume, and V' that of the cone. $V = \frac{4}{3}\pi \cdot R^3$, $V' = \pi \cdot R^2 \times 2R \times \frac{1}{3} = \frac{2}{3}\pi \cdot R^3$; $\therefore V : V' = 2 : 1$.

923. If R is the radius of the sphere, then $2R$ is a diagonal of the inscribed cube; $\therefore \frac{2}{\sqrt{3}}R =$ the edge of this cube; $\therefore$ if V, V' are the volumes, $V : V' = \frac{4}{3}\pi \cdot R^3 : \frac{8}{3\sqrt{3}}R^3 = \pi : \frac{2}{\sqrt{3}}$.

924. If R be the radius of the sphere, then $2R$ is an edge of the circumscribed cube; $\therefore V : V' = \frac{4}{3}\pi \cdot R^3 : 8R^3 = \pi : 6$.

APPENDIX.

Numbers in parentheses refer to exercises in the Appendix to Plane Geometry, bound separately.

925. (**644**) Suppose a circumference circumscribed about any regular polygon of $2n$ sides. The arcs subtended by any n successive sides form a semicircumference, the diameter of which passes through two opposite vertices, and is bisected at the center.

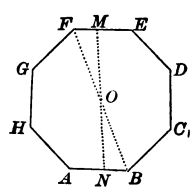
926. The diagonals — that is, the lines joining opposite vertices — of a parallelopiped bisect each other (Ex. 770).

927. (**645**) A circle is symmetrical with respect to its center (169).

928. (**646**) A parallelogram is symmetrical with respect to the point in which its diagonals intersect each other; for every intercept passing through that point is bisected there (145).

929. (**647**) If any two intercepts passing through a point were bisected in that point, the quadrilateral would be a parallelogram (Ex. 127), not a trapezium.

930. (**648**) Let $ABC \cdots H$ be a regular polygon of $2n$ sides, and BF a diameter joining two opposite vertices (Ex. 925). Since BF passes through the center O, BF bisects angles B and F. Through O draw any intercept cutting AB, EF, in N, M, respectively. Since $OB = OF$, $\angle BON = \angle FOM$, and $\angle ONB = \angle OMF$, $\triangle OBN = \triangle OFM$; $\therefore ON = OM$.



931. (**649**) The axis of symmetry of the extremities of a chord is the diameter drawn perpendicular to that chord (172).

932. (**650**) The axis of symmetry of two opposite vertices of a square is the diagonal joining the other two vertices.

933. (**651**) A circle has an indefinite number of axes of symmetry, seeing that each diameter is such an axis. Two circles have one common axis of symmetry, viz., their line of centers produced.

934. (**652**) An isosceles triangle is symmetrical with respect to the altitude drawn to its base.

935. (**653**) An equilateral triangle has three axes of symmetry, viz., the altitudes.

936. (**654**) (1) A square has four axes of symmetry; viz., the two diagonals and the two intercepts $\perp$ to the sides through the intersection of the diagonals. (2) A rhombus has two axes of symmetry; viz., the diagonals. (3) A regular pentagon has five axes of symmetry; viz., its five altitudes. (4) A regular hexagon has six axes of symmetry; viz., its three diagonals through opposite vertices, and the three intercepts perpendicular to opposite sides through the center. (5) A regular polygon of $2n$ sides has $2n$ axes of symmetry; viz., its n diagonals through opposite vertices, and the n intercepts perpendicular to opposite sides through the center. (6) A regular polygon of $2n+1$ sides has $2n+1$ axes of symmetry; viz., the $2n+1$ altitudes drawn from the vertices.

937. (**655**) AC is perpendicular to BD at its mid point (74). If from any point P in AC an intercept EPF be drawn parallel to BD, meeting CB, CD in E, F respectively, it is easily shown that EF is perpendicular to and bisected by AC.

938. Let AA' be a diagonal of a polyhedron having a center of symmetry O, and MM' be any intercept between opposite faces through O. Join AM, $A'M'$. The $\triangle\!\!\triangle\, AOM$, $A'OM'$ are in the same plane (423) and are equal (63), since $OA = OA'$ (Hyp.), $\angle MOA = \angle M'OA'$ (50), and $\angle OMA = \angle OM'A'$ (110); $\therefore\ OM = OM'$.

939. For (728) the bases must be polygons of an even number of sides parallel two and two, hence planes passed through opposite edges will be parallelograms (see diagram for 740); and, as those planes pass through the centers, O, O' of the bases, OO' is their common intersection, and diagonals such as BE' will pass through the mid point of OO', and this mid point is the center of symmetry (737).

940. The center of symmetry of a parallelopiped is the point in which the diagonals bisect each other (Ex. 770).

941. In the diagram for Art. 740, let AD' be a right prism whose bases have O and O' respectively as centers of symmetry. Pass the plane BE' through the opposite edges BB', EE'. Since BB' is $\parallel$ and $=$ to EE', BE is $\parallel$ and $=$ to $B'E'$ (142). Since BO is $\parallel$ and $=$ to $B'O'$

(Ax. 7), OO' is ∥ to BB'. But BB' is perpendicular to the bases (Hyp.); ∴ OO' is perpendicular to the bases, and hence to all the lines passing through, and bisected in, O and O'; ∴ OO' is an axis of symmetry.

942. A rectangular parallelopiped is a right prism, each of whose opposite pairs of faces may be taken as bases. Each of these bases, again, being a rectangle, is symmetrical with respect to a center. Hence (Ex. 941) there is an axis of symmetry for each of the three opposite pairs of faces.

943. Three, one for each opposite pair of faces.

944. For the base, being a regular polygon with an even number of sides, has a center of symmetry (Ex. 930), which is also the point at which the axis is perpendicular to the base. Hence the axis is a center of symmetry.

945. (1) If a right prism has irregular bases, it has but one plane of symmetry, the plane perpendicular to any of its edges at its mid point. For this plane will be perpendicular to, and bisect, all the edges.

(2) If a right prism has regular bases of m sides, it has $m + 1$ planes of symmetry; viz., that passed perpendicular to any edge at its mid point, and those passed through the m axes of symmetry of each base (Ex. 936, (5), (6)). For each of these planes divides the bases symmetrically, and hence also the prism.

946. A parallelopiped having rectangular bases, and each base, as may be easily shown, having four axes of symmetry, the parallelopiped has fifteen planes of symmetry; viz., 4×3 through the four axes of symmetry of each of the three pairs of opposite faces, and three equidistant from, and parallel to, each of the three pairs of opposite faces.

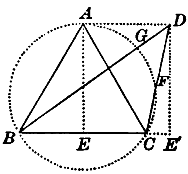
947. Any cylinder has a plane of symmetry equidistant from, and parallel to, its bases. A cylinder of revolution has, moreover, an indefinite number of planes of symmetry passing through its axis. Since any plane passed through its center bisects a sphere (593), the sphere has an indefinite number of planes of symmetry.

948. In the diagram for Art. 733, let HR be one of the bases of the polyhedron, and XX', YY' the intersections with HR of the two planes of symmetry perpendicular to each other. The planes being planes of symmetry, XX', YY' are axes of symmetry for HR, and being perpendicular to each other, their intersection O is a center of symmetry for HR (733). Similarly the point O' in base $H'R'$ is a center of symmetry for that base, and OO', the intersection of the planes of symmetry, is an axis of symmetry for the figure.

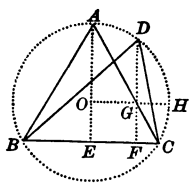
949. Since the planes are perpendicular to each, each intersection

is perpendicular to the plane of the other two (477), and is an axis of symmetry (Ex. 948); hence the point common to all three axes is a center of symmetry.

950. In order that they may form with the third side a triangle of maximum area, the two given lines must be perpendicular to each other (747). Let x be the numerical measure of the third line; then $x = \sqrt{a^2 + b^2}$.

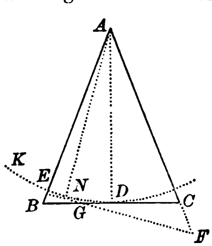


951. Let ABC, DBC be equivalent triangles upon the same base BC, ABC being isosceles. Draw the altitudes AE perpendicular to BC and DE' perpendicular to BC produced. $AE = DE'$, since the triangles are equivalent. Join AD; AD is parallel to BC and perpendicular to AE. Describe a circumference about $\triangle ABC$. Since AD is tangent to this circumference (190), D lies without it; and DB, DC must cut it in, say, G, F respectively. Since $\angle BAC$ is measured by $\frac{1}{2}$ arc BC, and $\angle BDC$ is measured by $\frac{1}{2}$ (arc BC − arc FG), $\angle BAC > \angle BDC$.



952. (**656**) Let ⊿s ABC, DBC have the same base BC, and equal vertical angles A and B, and ABC be isosceles. Describe a circumference about $\triangle ABC$; it will also circumscribe DBC, since they have equal vertical angles. Draw the altitudes AE and DF, and the radius $OGH \perp$ to AE and DG. Then $OE = GF$. But $AO > DG$ (184, Ax. 5); $\therefore$ $AO + OE$ or $AE > DG + GF$ or DF; $\therefore$ $\triangle ABC < \triangle DBC$ (333).

953. Let ABC be an isosceles triangle, having a given vertical $\angle A$ and a given altitude AD. From A as center, with radius AD, describe an arc of which KED is part. Any triangle, as AEF, having a vertical angle and altitude equal to those of ABC will, if applied to ABC so that the vertical angles coincide, have its base tangent to arc KED, and its altitude AN will fall on one side or the other of AD. Let it fall between AD and AB. Then EN will cut BC in some point G between D and B, and $GB < GC$. Since ANE is a right angle, $\angle AEN$ is acute; $\therefore$ $\angle BEG$ is obtuse; $\therefore$ $GE < GB < GC$. Since $\triangle ABC$ is isosceles, $\angle ACB$ is acute; $\therefore$ $\angle FCG$ is obtuse; $\therefore$ $FG > CG > GB$. Now $\triangle BGE : \triangle FGC = GB \cdot GE : GF \cdot GC$; $\therefore$ $\triangle BGE < \triangle FGC$; $\therefore$ adding $AEGC$, $\triangle ABC < \triangle AEF$.



954. In the diagram for Ex. 952, let ABC, DBC be the triangles, and join DE, DO, O being the center of the circumscribing circle. Then $\overline{AB}^2 + \overline{AC}^2 = 2\,\overline{AE}^2 + 2\,\overline{BE}^2$, and also $\overline{DB}^2 + \overline{DC}^2 = 2\,\overline{DE}^2 + 2\,\overline{BE}^2$ (353). But $\overline{AE}^2 > \overline{DE}^2$, since $AE = AO + OE = DG + OE > DE$;

$$\therefore\ \overline{AB}^2 + \overline{AC}^2 > \overline{DB}^2 + \overline{DC}^2.$$

$$\text{Now } 2\,AB \cdot AC > 2\,DB \cdot DC$$

$$(\text{since } \triangle ABC > \triangle DBC \text{ by Ex. 952});$$

$$\therefore\ (AB + AC)^2 > (DB + DC)^2;$$

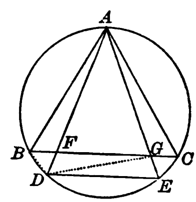
$$\therefore\ AB + AC > DB + DC;$$

$$\therefore\ AB + AC + BC > DB + DC + BC.$$

955. Let AB be the chord of the given arc ACB. Bisect ACB in C, and join CA, CB. ACB is an isosceles triangle (174); $\therefore CA + CB$ is greater than the sum, $PA + PB$, of the chords drawn from A and B to any other point P in the given arc (Ex. 954).

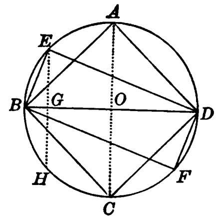
956. **(657)** In order that the triangle whose perimeter is n ft. may have a maximum area, it must be equilateral; hence each side will be $\frac{1}{3}n$ ft.; hence the area of the triangle $= \left(\frac{n}{6}\right)^2\sqrt{3}$ (368, 5°, Scholium).

957. **(658)** Let ABC be an inscribed equilateral triangle, and ADE any isosceles triangle inscribed in the same circle. If DE is not parallel to BC, it can be so placed. Join BD and DG. $\angle FDB$ is measured by $\frac{1}{2}$ arc AB, *i.e.*, by $\frac{1}{6}$ of a circumference, and $\angle FBD$ is measured by $\frac{1}{2}$ arc DC, which is less than $\frac{1}{6}$ of a circumference; $\therefore \angle FDB > \angle FBD$; $\therefore FB > FD$. Also $AD > AB$ (183); $\therefore AD > BC$; $\therefore AD - FD$ or $AF > BC - BF$ or $FC > FG$. But $\triangle AFB : \triangle GFD = AF \cdot FB : GF \cdot FD$; $\therefore \triangle AFB > \triangle GFD, > \frac{1}{2} FDEG$. Similarly $\triangle AGC > \frac{1}{2} FDEG$; $\therefore$ (adding AFG) $\triangle ABC > \triangle ADE$. But $\triangle ADE$ is greater than any scalene triangle of equal base and vertical angle (Ex. 952); $\therefore \triangle ABC$ is greater than any inscribed triangle having its base less than BC; and in a similar way it can be shown that $\triangle ABC$ is greater than any inscribed triangle having its base greater than BC.



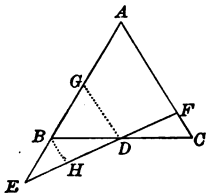
958. **(659)** In the diagram for Art. 327, let EC be a rectangle and AC an oblique parallelogram, having the same base BC and altitude BE; *i.e.*, having the same area. $BC + EF = BC + AD$; but as $EB < AB$ (93), $EB + FC < AB + DC$; $\therefore BC + EF + EB + FC < BC + AD + AB + DC$.

959. **(660)** If $AB > BC$, $(AB - BC)^2 > 0$; $\therefore 4\,\overline{AB}^2 + 4\,\overline{BC}^2 - 8\,AB \cdot BC > 0$; $\therefore 4\,\overline{AB}^2 + 4\,\overline{BC}^2 + 8\,AB \cdot BC > 16\,AB \cdot BC$; $\therefore 2\,(AB + BC) > 4\sqrt{AB \cdot BC}$. Now $2\,(AB + BC)$ is the perimeter of

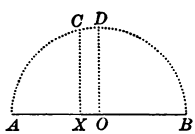


a rectangle contained by any sides AB, BC, and $4\sqrt{AB \cdot BC}$ is the perimeter of the equivalent square.

960. (**661**) Inscribed in $\odot\ O$, let $ABCD$ be a square, and $EBFD$ a rectangle, having the same diameter BD as diagonal. Join AC, and draw chord $EGH \perp$ to BD at G. Since $AC > EH$, $AO > EG$; $\therefore \triangle ABD > \triangle EBD$; $\therefore ABCD > EBFD$.



961. (**662**) Let ABC, AEF be two triangles having a common vertical angle A, and their bases BC, EF passing through a given point D, which is also the mid point of BC. Through D draw DG parallel to AC and meeting AB in G; and through B draw BH parallel to AC and meeting EF in H. As B lies between E and G, H lies between E and D. Then (63) $\triangle DCF = \triangle DBH$. But $\triangle DBH < \triangle DBE$; $\therefore \triangle DCF < \triangle DBE$; $\therefore$ (adding $ABDF$), $\triangle ABC < \triangle AEF$.

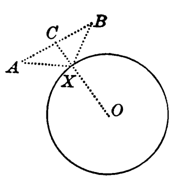


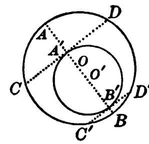
962. (**663**) Let AB be the given line, and upon it describe a semicircumference ACB. If at any point X in AB we erect a perpendicular to meet ACB in C, then rectangle $AX \cdot BX \simeq \overline{CX}^2$. But the perpendicular OD drawn at O, the mid point of AB, is greater than any other such perpendicular (184); hence $AO \cdot OB$ is the maximum rectangle.

963. (**664**) The maximum chord is the diameter through the given point (184); the minimum chord is that drawn through the point perpendicular to the maximum chord (Ex. 199).

964. (**665**) The secant drawn from the given point through the center has its outer segment a minimum and its inner segment a maximum (see solution of Ex. 172).

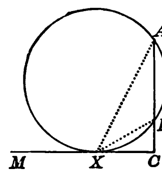
965. (**666**) Let A and B be the given points without $\odot\ O$. Join AB, and from the mid point C draw CO, cutting the circumference in X. X is the required point. Join AX, BX. $\overline{AX}^2 + \overline{BX}^2 \simeq 2\,\overline{CX}^2 + 2\,\overline{AC}^2$. Now to whatever point in the circumference lines be drawn from A and B, the sum of the squares of these lines is equal to the sum of twice the square of the median plus twice $\overline{AC}^2$. But AC is constant, hence we have to find the required point so that the median shall be a minimum. Now of all secants drawn from C, the minimum is that





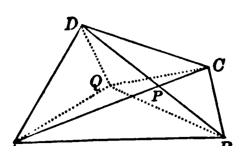
which passes through the center (Ex. 964). Hence $\overline{AX}^2 + \overline{BX}^2$ is a minimum.

966. (**667**) Let $\odot\, O'$ lie entirely within $\odot\, O$, but not be concentric with it. Join OO', and produce both ways to meet circumference of $\odot\, O$ in A, B, and that of $\odot\, O'$ in A', B'. Draw chords CD, $C'D'$ of $\odot\, O$ tangent to $\odot\, O'$ at A', B' respectively. Let O lie between O' and A'; OA' is the least, and OB' the greatest, line that can be drawn from O to the circumference of $\odot\, O'$ (Ex. 171). Hence the chord at A' is the maximum, and that at B' the minimum, chord of $\odot\, O$ that can be drawn touching $\odot\, O'$.



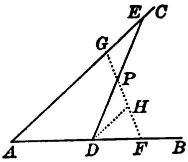
967. (**668**) Let AB produced meet CM in C. Through A and B describe a circumference touching CM in X. To do this, through D, the mid point of AB, draw $DO \perp$ to AB; then with center A and radius equal to DC, describe an arc cutting DO in O; O is the center of the required circle. Since X is the only point in CM that does not lie without the circumference, $\angle AXB$ is greater than any angle formed by lines drawn from A and B to any other point in CM. For $\angle AXB$ is measured by $\frac{1}{2}$ arc AB, but any other angle APB, having its vertex at some other point P in CM, would be measured by $\frac{1}{2}$ (arc AB − arc PB) (271).

REMARK.—This problem may be stated in a more general form by not supposing A and B to be perpendicular to CM, and is proved in a similar way. In the form above given, this problem affords an easy method of finding the distance from which an elevated object, a flagstaff AB on a building BC, for example, will be seen under the greatest visual angle, if the lengths AB and BC are known. Find a mean proportional between AC and BC; this will give CX (303), care being taken first to subtract from BC the height of the observer's eye. Thus suppose the height BC of a schoolhouse is 50 ft., the flagstaff AB is 20 ft., and the height of the observer's eye is 5 ft. Then $\sqrt{(45+20)\times 45} = 54.08$, gives 54 ft., about, as the distance from the foot of AC at which AB will seem greatest.



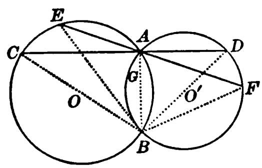
968. (**669**) Let $ABCD$ be the quadrilateral, and AC, BD its diagonals intersecting in P. P is the required point. For take any other point Q, and join QA, QB, QC, and QD. Then $QA + QC > AC > PA + PC$; etc.

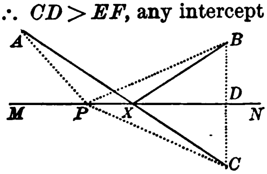
969. (**670**) Let BAC be the given angle, and P the given point within it. Through P draw the intercept DE, bisected in P (219). DE is the required line. For let FG be any other



intercept passing through P. From D draw $DH \parallel$ to AC, and meeting FG in H. $\triangle PDH = \triangle PEG$ (63). But $\triangle PDF > \triangle PDH$; $\therefore \triangle PDF > \triangle PEG$; $\therefore ADPG + \triangle PDF > ADPG + \triangle PEG$; *i.e.*, $\triangle GAF > \triangle EAD$.

970. (**671**) Let A be a point of intersection of ⓈO and O'. Draw the common chord AB, and through A draw $CD \perp$ to AB, and meeting the circumferences in C and D. CD is the required line. For through A draw any other intercept EF, and join B with C, E, D, and F. Since BAC, BAD are right angles (Const.), BC and BD are diameters. Since (302) $\angle C$ is measured by $\frac{1}{2}$(arc BFD − arc BGA) = $\frac{1}{2}$(arc BAD − arc BGA) = $\frac{1}{2}$ arc AD, and $\angle ABD$ is also measured by $\frac{1}{2}$ arc AD, $\angle C = \angle ABD$. But $\angle ABC$ is complement of $\angle C$, hence also of $\angle ABD$; $\therefore \angle CBD$ is a right angle; and, since $\angle E = \angle C$, being measured by $\frac{1}{2}$ arc AB of $\odot O$ and $\angle F = \angle D$ (266), $\angle EBF$ is also a right angle. Hence $\overline{CD}^2 \approx \overline{BC}^2 + \overline{BD}^2$, and $\overline{EF}^2 \approx \overline{BE}^2 + \overline{BF}^2$; but $BC > BE$ and $BD > BF$ (184); $\therefore \overline{CD}^2 > \overline{EF}^2$; $\therefore CD > EF$, any intercept not perpendicular to AB.





971. (**672**) Let A and B be the given points on the same side of the given line MN. Draw $BD \perp$ to MN, and produce to C, so that $DC = BD$. Draw AC, cutting MN in X; then X, the point at which lines drawn from A and B make equal angles with MN (Ex. 248), is also the point such that the sum of the distances from it to A and B is a minimum. For join A, B, and C with P, any other point in MN, and join BX. Since X and P are in the perpendicular at the mid point of BC (Const.), $XC = XB$ and $PC = PB$; $\therefore AC = AX + BX$, and $AP + PC = AP + PB$. But $AC < AP + PC$; $\therefore AX + BX < AP + BP$.

Remark.—In optics it is known that if AX be a ray of light reflected from a mirror MN at X, $\angle BXN = \angle AXM$, and from this proposition we find that the path between two points, A and B, of such a reflected ray is the shortest possible.

972. (**673**) The required point is found by producing a radius of the given circle so as to be perpendicular to the given line (Ex. 422).